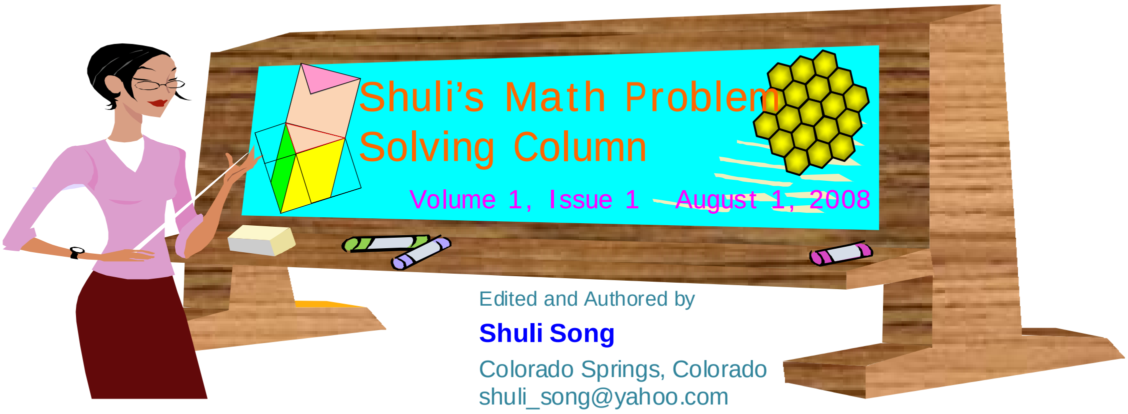

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Content

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Basic Knowledge

Choose *m* Elements from *n*

This short lesson will answer the following questions:

Choose 5 people from 10 to play basketball. How many ways are there?

There are 6 different dishes in a dinner. How many ways are there to choose 3 dishes?

In the following context *m*, *n*, etc. are positive integers if they are not specified otherwise.

In general, how many ways are there to choose m elements from n $(n \ge m)$?

In all of these problems the order of selected elements does not matter.

In combinatorics the number of ways to choose m elements from *n* is denoted by $\binom{n}{m}$, which is equal to

$$\frac{n\cdot(n-1)\cdot\cdots\cdot(n-m+1)}{m\cdot(m-1)\cdot\cdots\cdot 1}.$$

Why is it?

We can obtain the answer as follows:

There are *n* choices to select the first element. After the first element is selected, there are $n-1$ choices ($n-1$ elements left) to select the second element. After the first and second elements are selected, there are $n-2$ choices ($n-2$ elements left) to select the third element. Continuing this we have $n-(m-1)=n-m+1$ choices to select the last element. Therefore, there are $n\cdot(n-1)\cdot\cdots\cdot(n-m+1)$ ways to select *m* elements.

However, the same set of *m* elements is counted more than once. We can use the same methodology to obtain the number of times one set of *m* elements is counted. It is equal to $m\cdot(m-1)\cdot\cdots\cdot 1$. Therefore,

$$\binom{n}{m}=\frac{n\cdot(n-1)\cdot\cdots\cdot(n-m+1)}{m\cdot(m-1)\cdot\cdots\cdot 1}.$$

We introduce the factorial of *m*, which is defined as the product of all *m* positive integers from *m* down to 1:

$$m! = m\cdot(m-1)\cdot\cdots\cdot 1.$$

Note $n! = n\cdot(n-1)\cdot\cdots\cdot(n-m+1)\cdot(n-m)!$. We have

$$\binom{n}{m}=\frac{n!}{m!\cdot(n-m)!}.$$

To be concrete,

$$\binom{5}{2}=\frac{5\cdot 4}{2\cdot 1}=10 \text{ and } \binom{7}{4}=\frac{7\cdot 6\cdot 5\cdot 4}{4\cdot 3\cdot 2\cdot 1}=35.$$

The following formula is often useful:

$$\binom{n}{m}=\binom{n}{n-m}.$$

The formula is obvious because choosing m elements from n to go is the same thing as choosing $n-m$ elements from *n* to stay.

For $\binom{10}{7}$ we often calculate $\binom{10}{3}=\frac{10\cdot 9\cdot 8}{3\cdot 2\cdot 1}=120$.

Note that $\binom{n}{1}=\binom{n}{n-1}=n$.

What is $\binom{n}{0}$?

There is only one possibility to choose nothing from n elements, that is, not to choose anything. So $\binom{n}{0}=1$.

Practice Problems

1. Calculate for each:

$$\binom{8}{5} \quad \binom{5}{4} \quad \binom{100}{2} \quad \binom{10}{4} \quad \binom{7}{5} \quad \binom{12}{3}$$

2. In a room there are 6 people, how many ways are there to choose 3 people to clean the room?
3. In a party there are 10 people. Any two people make one and only one handshake. How many handshakes are made altogether?
4. In a bridge club there are 12 members. How many ways are there to form a team of 4 people?
5. How many ways are there to choose three different numbers from the set $\{1, 2, 3, 4, 5, 6, 7, 8, 9\}$?
6. How many ways are there to choose three different odd numbers from the set $\{1, 2, 3, \cdots, 100\}$?
7. How many ways are there to choose three different numbers from the set $\{1, 2, 3, \cdots, 100\}$, whose sum is odd?
8. No three diagonals of a convex dodecagon (12-sided polygon) intersect at one point inside the polygon. How many intersection points of the diagonals are there inside the polygon?

Math Competition Skill

Model I of Balls and Sticks

Problem and Solution

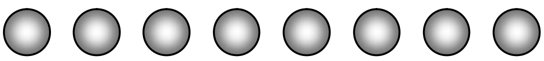
Problem 1

Five boxes are numbered 1 through 5. How many ways are there to put 8 identical balls into these boxes such that none of them is empty?

Answer: $\binom{7}{4}=35$

Solution:

Arrange 8 balls in a row:

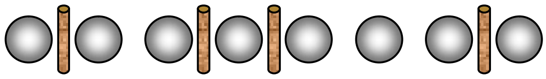


To determine a distribution of 8 balls in 5 boxes, we may partition this row of balls into 5 groups with 4 sticks.

Look at the five numbers, which are the number of balls left to the leftmost stick, the three numbers of balls between the four sticks, and the number of balls right to the rightmost stick. No number can be 0.

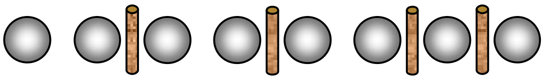
One partition corresponds to one distribution of 8 balls in 5 boxes, and vice versa.

For example, partition



corresponds to distribution 1, 2, 1, 3, 1, which are the numbers of balls in boxes 1 to 5 respectively.

Distribution 2, 2, 2, 1, 1 corresponds to partition



Now the problem becomes: how many ways are there to partition 8 balls with 4 sticks?

Between 8 balls there are 7 gaps. There are $\binom{7}{4}$ ways to choose 4 gaps for the 4 sticks.

This yields the answer to the problem.

Remarks

1. In this problem there is no empty box. Pay attention to this condition in a similar problem.
2. The boxes are different (numbered with different numbers). This means that distribution 1, 2, 3, 1, 1 is different from distribution 1, 1, 1, 2, 3.
3. The problem can be written in the general way:

   m boxes are numbered 1 through m. How many ways are there to put n $(n \ge m)$ identical balls into these boxes such that none of them is empty?

   Arrange n balls in a row. Between n balls there are $n-1$ gaps. Put $m-1$ sticks to separate them into m groups. The answer is $\binom{n-1}{m-1}$.
4. In some problems we may have to figure out what are balls, and what are sticks. The following examples may give some ideas.

Similar Problems

Problem 2

How many ways are there to express 8 as the sum of 5 positive integers if the order of numbers in an expression is counted?

Answer: $\binom{7}{4}=35$

Solution:

Is this the same problem as Problem 1? We will see.

Place 8 ones in a row:

1 1 1 1 1 1 1 1

Arrange 4 "+" signs in the 7 gaps between 8 ones. One arrangement of 4 "+" signs between 8 ones corresponds to one expression required.

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For example, arrangement

$$\boxed{1} + \boxed{1 \quad 1} + \boxed{1} + \boxed{1 \quad 1 \quad 1} + \boxed{1}$$

indicates expression

$$1+2+1+3+1.$$

Expression

$$2+2+2+1+1$$

corresponds to arrangement

$$\boxed{1 \quad 1} + \boxed{1 \quad 1} + \boxed{1 \quad 1} + \boxed{1} + \boxed{1}$$

There are $\binom{7}{4}$ ways to put the 4 "+" signs into the 7 gaps.

This is the answer to the problem.

In fact, this is the same problem as Problem 1.

Problem 3

A bookbinder has to bind 10 identical books using red, green, yellow, or blue covers. In how many ways can he do this if there is at least one book covered by each of the four colors?

Answer: $\binom{9}{3}=84$

Solution:

Partition 10 books in a row into four groups for the red, green, yellow, and blue covers respectively. There are $\binom{9}{3}$ ways to do this.

Problem 4

Thirty people vote for 5 candidates. How many possible distributions of their votes are there, if each of the 30 people votes for one candidate only, and each candidate gets at least one vote? Consider only the numbers of votes to candidates. That is, all of the votes are the same.

Answer: $\binom{29}{4}=23{,}751$

Solution:

Partition 30 votes in a row into 5 groups for the 5 candidates respectively. There are $\binom{29}{4}$ ways to do this.

Problem 5

There are 6 types of postcards in a post office. How many ways are there to buy 15 postcards if there is at least one postcard of each type?

Answer: $\binom{14}{5}=2002$

Solution:

Partition 15 postcards in a row into 6 groups for the 6 types respectively. There are $\binom{14}{5}$ ways to do this.

Practice Problems

1. How many ways are there to express 15 as the sum of 4 positive integers if the order of numbers in an expression is counted?
2. A bookbinder has to bind 9 identical books using red, green, yellow, black, or blue covers. In how many ways can he do this if there is at least one book covered by each of the five colors?
3. Forty people vote for 7 candidates. How many possible distributions of their votes are there, if each of the 40 people votes for one candidate only, and each candidate gets at least one vote? Consider only the numbers of votes to candidates. That is, all of the votes are the same.
4. There are 8 types of postcards in a post office. How many ways are there to buy 20 postcards if there is at least one postcard of each type?
5. In how many ways can 12 pennies be put into 5 different purses such that none of them is empty?
6. How many ways are there to cut an open necklace with 30 pearls into 8 parts if every part must have at least one pearl?
7. A train with n passengers is going to make m $(n \ge m)$ stops. How many ways are there for passengers to get off the train at the stops if we take into account only the number of passengers who get off at each stop, and at each stop there is at least one passenger who gets off?
8. In how many ways can 3 people divide 6 apples and 8 pears if each person gets at least one fruit of each kind?
9. How many ways are there to put 8 red, 8 blue, and 8 green balls into 4 different boxes if each box has at least one ball of each color?
10. How many ordered triples of positive integers (a,b,c) are there such that $a+b+c=20$?

A Problem from a Real Math Competition

Today's problem comes from American Mathematics Contest Grade 8 (AMC8).

Problem

(20^{th} AMC8 2004 Problem 17)

Three friends have a total of 6 identical pencils, and each one has at least one pencil. In how many ways can this happen?

Answer: 10

Solution One (Systematically Listing):

Systematically listing works since the numbers in this problem are not large.

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Let A, B, and C be the three friends.

We list the numbers of pencils from the largest to the smallest systematically.

A	*B*	*C*
4	1	1
3	2	1
3	1	2
2	3	1
2	2	2
2	1	3
1	4	1
1	3	2
1	2	3
1	1	4

We observe 10 ways to distribute 6 pencils.

Solution Two (Systematically Listing):

We will still systematically list but in an easier way.

There are three cases to distribute 6 pencils.

Case 1: 4 pencils + 1 pencil + 1 pencil

Any one of three friends may have four pencils. Each of the rest has one pencil.

In this case there are 3 possibilities.

Case 2: 3 pencils + 2 pencils + 1 pencil

Any one of three friends may have three pencils, any one of the rest may have two pencils, and the last has one pencil.

There exist $3 \cdot 2 \cdot 1 = 6$ possibilities in this case.

Case 3: 2 pencils + 2 pencils + 2 pencils

Each of three friends has two pencils.

So only one possibility exists in this case.

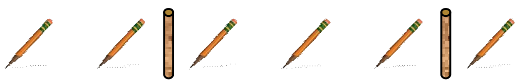
Therefore, the answer is $3+6+1=10$.

Solution Three (Model I of Balls and Sticks):

Model I of Balls and Sticks, which we just learnt, solves the problem.

Arrange 6 pencils in a row:

To determine a distribution of 6 pencils to 3 friends, we may partition this row into 3 groups of pencils with 2 sticks:



The number of ways to distribute 6 pencils to 3 friends is equal to the number of ways to partition 6 pencils into 3 groups with 2 sticks.

Note that there are 5 gaps between 6 pencils. There are $\binom{5}{2}$ ways to choose 2 gaps from 5 for the 2 sticks.

Therefore, the answer is $\binom{5}{2}=10$.

Practice Problem

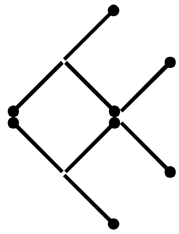
(MathCounts 2008 State Target Problem 2)

Bill is sent to a donut shop to purchase exactly six donuts. If the shop has four kinds of donuts and Bill is to get at least one of each kind, how many combinations will satisfy Bill's order requirements?

Creative Thinking Problems 1 to 3

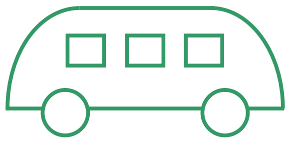
1. Swimming Fish

The fish below is swimming left. Move two matchsticks such that the fish changes direction.



2. Moving Bus

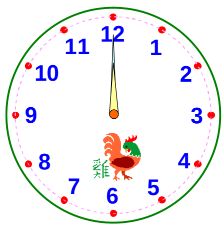
Look carefully at the bus in the figure below. Is the bus going left or right? How do you know?



3. A Careless Clockmaker

A clockmaker made a big mistake. He created a clock by placing the minute hand in the hour hand position and the hour hand in the minute hand position. Therefore, this clock shows the time going much faster with much more frequent hour changes.

The clockmaker initially set the time correctly at twelve o'clock.



Of course, the clock would show the time incorrectly after that. However, once in a while the clock does show the time correctly.

Figure out how many times, and at what instances, the clock correctly shows the time during a 12-hour time period.

(Solutions will be presented in the next issue.)

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