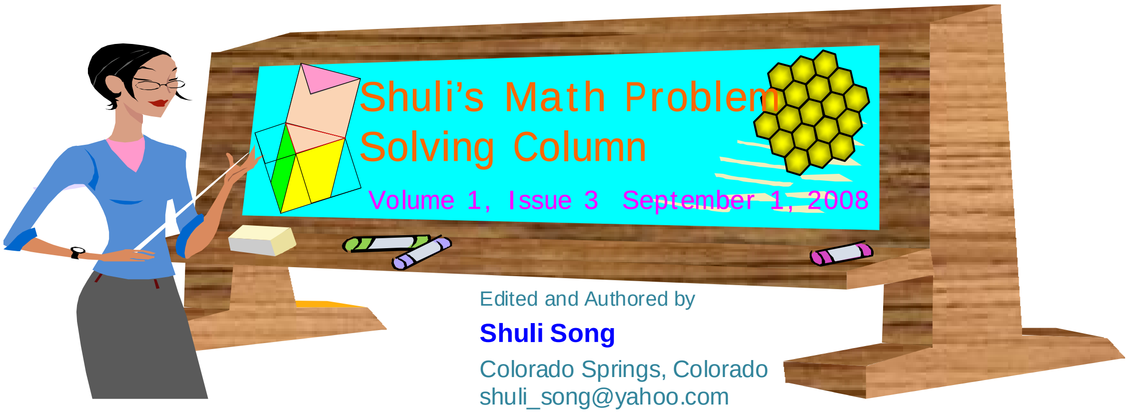

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Content

Basic Knowledge

Last Number You Count

Continuing the last issue, this short lesson will answer the following questions:

If you count 40 numbers from 50 continuously, what is the last number you will count?

If you count 35 numbers by 2's from 56, what is the last number you will count?

If you count 30 numbers by 6's from 1, what is the last number you will count?

The problem can be written in the general way:

If you count c numbers by k's from m where c is a positive integer, what is the last number you will count?

The answer is:

$$(c-1)\cdot k+m\ .$$

The problem is reversed in the "Basic Knowledge" section of the last issue.

There are also three steps to obtain the answer. Going backwards, we have three steps:

1. Subtract 1 from c.
2. Multiply the result by the number by which you count.
3. Add the product to the first number.

In the first question $m=50$, $c=40$, and $k=1$. So the answer is $(40-1)\cdot 1+50=89$.

Note that the answer is not $40+50=90$.

In the second $m=56$, $c=35$, and $k=2$. The answer is $(35-1)\cdot 2+56=124$.

In the third $m=1$, $c=30$, and $k=6$. The answer is $(30-1)\cdot 6+1=175$.

Why do we subtract 1 from c in the first step?

c is the number of numbers. In the calculation we need the number of intervals from the first number to the last. The number of intervals is one less than the number of numbers.

Practice Problems

1. If you count 100 numbers continuously from 10, what is the last number you will count?
2. If you count 28 numbers by 3's from 101, what is the last number you will count?
3. If you count 9 numbers by 9's from 9, what is the last number you will count?
4. If you count 888 numbers by 8's from 88, what is the last number you will count?
5. If you count 123 numbers by 456's from 789, what is the last number you will count?
6. If you count 29 numbers by 4's from 1896, what is the last number you will count?
7. There are 100 houses on a street. The first house is numbered 4321. The number difference between any two neighboring houses is 8. What is the number of the last house?
8. A boy was born on February 29, 1988. So he has a birthday every four years. In which year will he have his 20^{th} birthday?

Math Competition Skill

How Do You Count? – Using a Reference

Problem

Figure 1 shows a T-shaped tetromino.

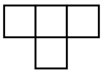


Figure 1: T-Shaped Tetromino

Figure 2 is an 8×8 grid.

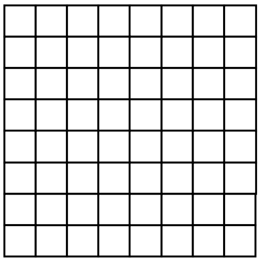


Figure 2: 8 × 8 Grid

How many ways are there to cut off a T-shaped tetromino along the grid lines from the 8×8 grid?

Solution

This lesson will introduce a method, which is called *Using a Reference*, for counting problems. Instead of counting the original objects, we will count by considering the reference. The reference should be simpler in shape and can be more easily managed.

We may take the red square shown in Figure 3 as the reference. Call this square the *head square*.

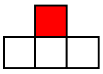


Figure 3: Head Square

Now consider how many ways we can place the head square in the grid.

We categorize the grid squares into three kinds: *corner*, *side*, and *inner squares*. In Figure 4, the 4 corner squares are colored green, the $4\times 6=24$ side squares are colored blue, and the $6\times 6=36$ inner squares are colored orange.

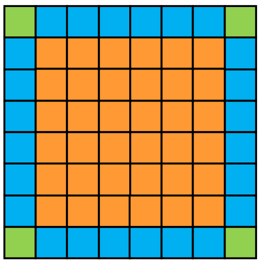


Figure 4: Three Kids of Squares

We will obtain the answer by recognizing the differences in these three kinds of squares.

We cannot put the head square at any of the four corner squares in the grid. We can put it at any of other squares. However, putting the head square at a side square is different from putting the head square at an inner square

Put the head square at a side square. The orientation of the tetromino is fixed, shown in Figure 5. So there is only one way to put the head square at any side square.

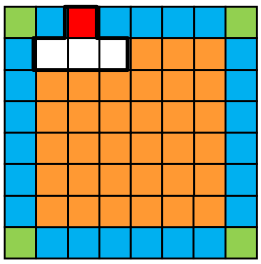


Figure 5: Head Square at a Side Square

Put the head square at an inner square. The tetromino can be rotated, shown in Figure 6. So there are four possibilities to put the head square at any inner square.

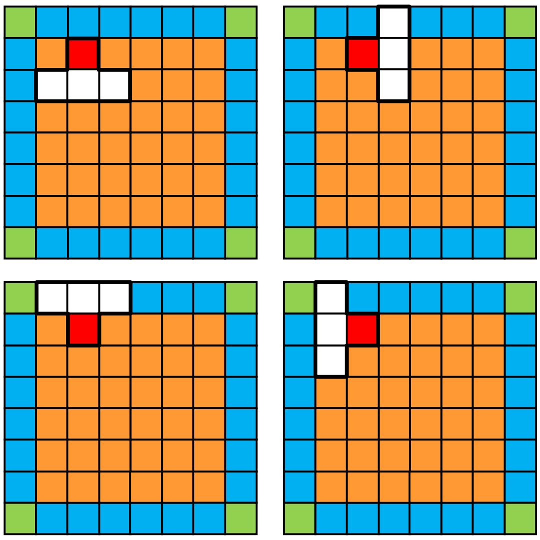


Figure 6: Head Square at an Inner Square

Now we see the answer $0\cdot 4+1\cdot 24+4\times 36=168$.

In this problem we may use the pink square in Figure 7 as the reference. The analysis will be the same.

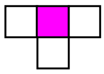


Figure 7: Another Possible Reference

It may not work to use a yellow square in Figure 8 as the reference because the yellow square is not unique.

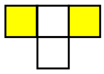


Figure 8: Not a Good Reference

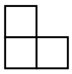
So choosing a proper reference is the key in using this method to solve a problem.

Practice Problems I

1. How many ways are there to cut off a T-shaped tetromino from a 6×6 grid?
2. How many ways are there to cut off a T-shaped tetromino from a 7×9 grid?
3. How many ways are there to cut off a T-shaped tetromino from an $n\times n$ grid, where $n\geq 3$?
4. How many ways are there to cut off a T-shaped tetromino from an $m\times n$ grid, where $m, n\geq 3$?

Practice Problems II

The figure below shows an L-shaped tromino.



1. How many ways are there to cut off an L-shaped tromino from a 6×6 grid?
2. How many ways are there to cut off an L-shaped tromino from a 7×9 grid?
3. How many ways are there to cut off an L-shaped tromino from a 8×8 grid?

A Problem from a Real Math Competition

Today's problem comes from the University of Northern Colorado Mathematics Contest (UNCMC).

Problem

(UNCMC 2005-2006 First Round Problem 11)

How many triples of positive integers *a*, *b*, *c*, are there with $a<b<c$ such that $a+b+c=21$?

Answer: 27

Solution One:

For a small number like 21, simply listing works well.

All triples for *a*, *b*, *c* are listed below:

a	b	c	$a+b+c$
1	2	18	21
1	3	17	21
⋮	⋮	⋮	⋮
1	9	11	21
2	3	16	21
2	4	15	21
⋮	⋮	⋮	⋮
2	9	10	21
3	4	14	21
3	5	13	21
⋮	⋮	⋮	⋮
3	8	10	21
4	5	12	21
4	6	11	21
4	7	10	21
4	8	9	21
5	6	10	21
5	7	9	21
6	7	8	21

The number of the triples is $8+7+5+4+2+1=27$.

Solution Two:

First we ignore the condition $a<b<c$. We count the number of triples of *a*, *b*, *c* in any order.

We place 21 ones in a row, and we will try to put two +'s in the 20 gaps between those 21 ones.

$$1\quad 1\quad 1\ +\ 1\quad 1\quad \cdots\quad 1\quad 1\ +\ 1$$

Let *a* be the number of ones before the first +, *b* be the number of ones between two +'s, and *c* be the number of ones after the second +. For example, the arrangement above corresponds to $a=3$, $b=17$, and $c=1$.

We have expression $a+b+c=21$.

The number of ways we have to put two +'s is equal to the number of triples *a*, *b*, *c* such that $a+b+c=21$.

The number of ways to choose two gaps from 20 gaps for the two +'s is $\binom{20}{2}=190$.

Now we go back to the original problem: the order of the three numbers does not matter.

Among 190 triples, there are 10 sets in which two or more of three numbers are the same, that is, $\{1,1,19\}$, $\{2,2,17\}$, $\{3,3,15\}$, $\{4,4,13\}$, $\{5,5,11\}$, $\{6,6,9\}$, $\{7,7,7\}$, $\{8,8,5\}$, $\{9,9,3\}$, $\{10,10,1\}$.

Nine sets are counted three times except $\{7,7,7\}$.

Among 190 triples, there are $190-3\times 9-1=162$ triples in which a, b, c are all different.

There are $3!=6$ orders for every set of three different numbers. Therefore, the answer is $\frac{162}{6}=27$.

Practice Problem

(UNCMC 2005-2006 Final Round Problem 10)

How many triples of positive integers *a*, *b*, and *c* are there with $a<b<c$ such that $a+b+c=401$?

Answers to All Practice Problems in Last Issue

How Many Numbers You Count

1. 334
2. 40
3. 251
4. 90
5. 900
6. 643

Model II of Balls and Sticks

1. $\binom{18}{3}=816$
2. $\binom{16}{4}=1820$
3. $\binom{46}{6}=9{,}366{,}819$
4. $\binom{12}{4}=495$
5. $\binom{16}{4}=1820$
6. $\binom{37}{7}=10{,}295{,}472$
7. $\binom{n+m-1}{n}$
8. $\binom{8}{2}\binom{9}{2}\binom{10}{2}=45{,}360$
9. $\binom{11}{3}\binom{11}{3}\binom{11}{3}=4{,}492{,}125$
10. $\binom{22}{2}=231$

A Problem from a Real Math Competition

1200

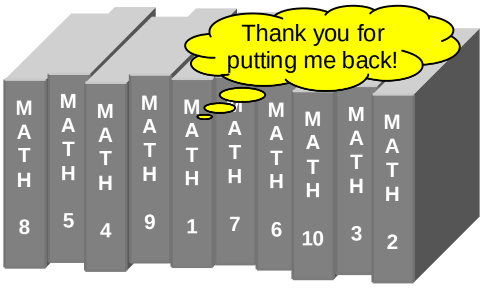
Solutions to Creative Thinking Problems 4 to 6

4. Sorting Books

How do people often sort their files?

Alphabetically!!!

The books are sorted alphabetically as well according to the numbers: *Eight*, *Five*, *Four*, *Nine*, $\cdots$. So Volume *One* is between Volume *Nine* and Volume *Seven*.



5. Four Liters of Water

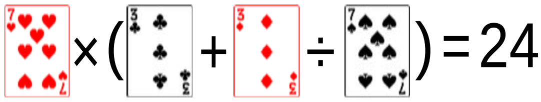
Note that $2\times5-2\times3=4$. So we will have 4 liters of water if we get water twice with the 5-liter container and pour out water twice with the 3-liter container. Then we have the following process:

Steps	*3-Liter*	*5-Liter*	
Start	0	0	
1st	0	5	5-liter water in (1st time)
2nd	3	2	
3rd	0	2	3-liter water out (1st time)
4th	2	0	
5th	2	5	5-liter water in (2nd time)
6th	3	4	
7th	0	4	3-liter water out (2nd time)

After the 7 steps, 4 liters of water is left.

Also note that $3\times3-1\times5=4$. We can have a different process. You are encouraged to find it.

6. Make 24 with 3, 3, 7, and 7



Creative Thinking Problems 7 to 9

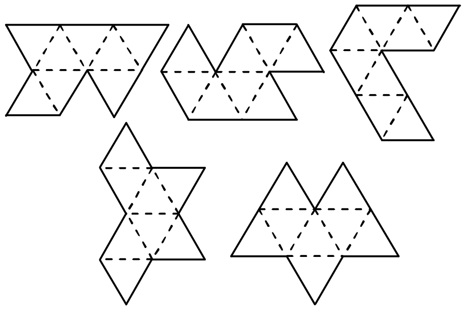
7. Odd Man

Which is the odd man (the least like the others)?

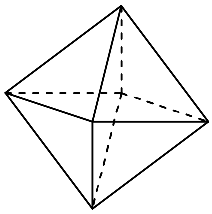
A) **CD** B) **ND** C) **RD** D) **ST** E) **TH**

8. Display of an Octahedron

Which of the following five figures



can be folded along its dashed lines to form a regular octahedron shown below?



9. Same Time and Same Place

An old man started climbing a mountain from the base at 8:00 am and reached the top at 8:00 pm. On the second day he started descending the mountain at 8:00 am along the same route and reached the base at 8:00 pm.

Prove that there was a time instance at which the old man was at the same route point on the mountain during the two days.

(Solutions will be presented in the next issue.)

