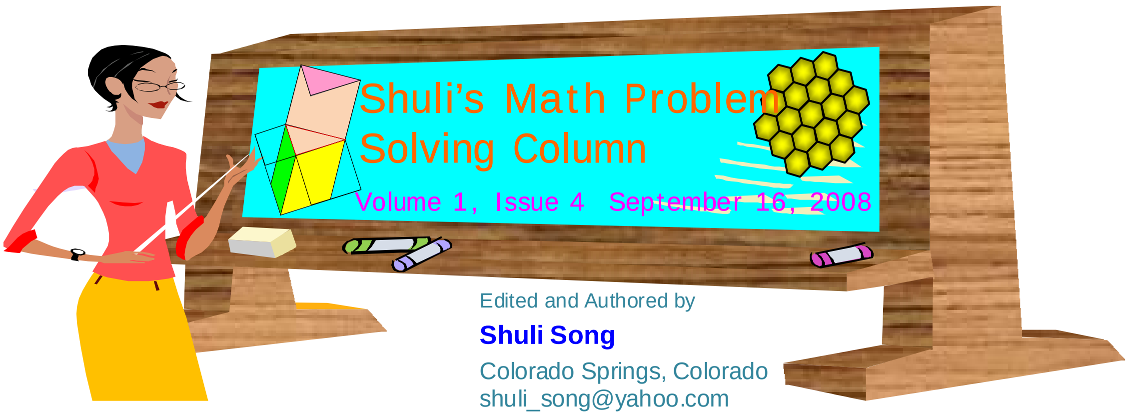


Content

Math Trick

Mental Calculation: $\overline{ab}\times\overline{ac}$

The Trick

Can you obtain the product for each in 3 seconds?

$52\times58=$ $\quad$ $31\times39=$ $\quad$ $67\times63=$

$44\times46=$ $\quad$ $85\times85=$ $\quad$ $94\times96=$

It may be impossible if you don't know the trick.

There are two properties in common for all these pairs of two-digit numbers:

1. The sum of the two ones digits is 10.
2. The two tens digits are the same.

We write the multiplications in the general form: $\overline{ab}\times\overline{ac}$ where *a*, *b*, and *c* are digits with $b+c=10$.

Following the given steps below you will be able to complete one multiplication in 3 seconds.

Example 1

Calculate 52×58.

Step 1: Calculate $a(a+1)$. In this example, $5\times6=30$.

Step 2: Attach $b\times c$ as a **TWO**-digit number to the right of the result obtained in step 1. In this example, attach $2\times8=16$ to the right of 30: 3016.

Now we are done: $52\times58=3016$.

Example 2

Calculate 31×39.

Step 1: Calculate $3\times4=12$

Step 2: Attach $1\times9=9$, treated as two digits 09: 1209

Then $31\times39=1209$.

The trick works for a multiplication of two numbers with three or more digits as well.

Example 3

Calculate 254×256

Step 1: Calculate $25\times26=650$

To calculate 25×26 we may use the trick again: $25\times26=25\times25+25=625+25=650$.

Step 2: Attach $4\times6=24$: 65024

So $254\times256=65024$.

Why Does This Work?

Write $\overline{ab}$ and $\overline{ac}$ in the base 10 representation:

$$\overline{ab}=10a+b \text{ and } \overline{ac}=10a+c.$$

Then we have

$$\overline{ab}\times\overline{ac}=(10a+b)(10a+c)=100a^2+10ac+10ab+bc$$
$$=100a^2+10a(b+c)+bc.$$

Noting that $b+c=10$ we obtain

$$\overline{ab}\times\overline{ac}=100a^2+100a+bc=100a(a+1)+bc.$$

This shows that to calculate $\overline{ab}\times\overline{ac}$ we may calculate $a(a+1)$, multiply the result by 100, and add bc.

So we can obtain the product by calculating $a(a+1)$ and attaching $b\times c$ as a two-digit number to the right of $a(a+1)$.

Practice Problems I

$84\times 86=$	$75\times 75=$	$53\times 57=$
$27\times 23=$	$91\times 99=$	$48\times 42=$
$69\times 61=$	$12\times 18=$	$45\times 45=$
$63\times 67=$	$88\times 82=$	$54\times 56=$

Practice Problems II

$107\times 103=$	$458\times 452=$	$994\times 996=$
$891\times 899=$	$498\times 492=$	$9993\times 9997=$

Math Competition Skill

How Do You Count? – Moving the Shape

Problem

We have the same problem in this issue as in the last.

Figure 1 shows a T-shaped tetromino.

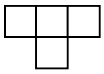


Figure 1: T-Shaped Tetromino

Figure 2 is an 8×8 grid.

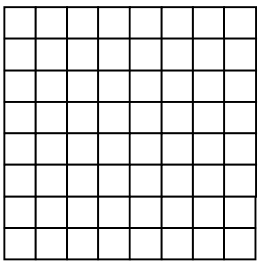


Figure 2: 8 × 8 Grid

How many ways are there to cut off a T-shaped tetromino along the grid lines from the 8×8 grid?

Solution

A new method will be introduced for counting problems, which is called *Moving the Shape*.

A T-shaped tetromino has four different orientations shown in Figure 3.

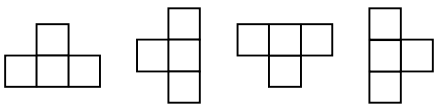


Figure 3: Four Orientations of a T-Shaped Tetromino

Consider the first orientation.

Place the tetromino at the top-left corner. Then see how many different positions there are to move the tetromino in the grid.

Counting the top-left corner position as the first position, we have 6 positions to move the tetromino right to the top-right corner.

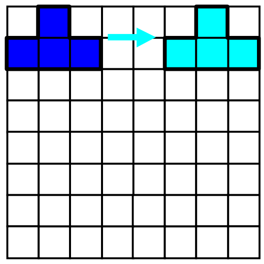


Figure 4: Moving the Shape Horizontally

We have 7 positions to move the tetromino down to the bottom-left corner.

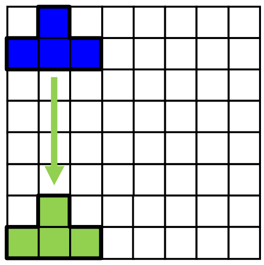


Figure 5: Moving the Shape Vertically

So we have $6\times 7=42$ positions to move the tetromino in the grid. That is, there are 42 ways to cut off a T-shaped tetromino in the first orientation.

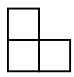
Since the four orientations are symmetrical, the total number of ways to cut off a T-shaped tetromino is $42\times 4=168$ as the answer.

Practice Problems I

1. How many ways are there to cut off a T-shaped tetromino from a 6×6 grid?
2. How many ways are there to cut off a T-shaped tetromino from a 7×9 grid?
3. How many ways are there to cut off a T-shaped tetromino from an $n\times n$ grid, where $n\geq 3$?
4. How many ways are there to cut off a T-shaped tetromino from an $m\times n$ grid, where $m,n\geq 3$?

Practice Problems II

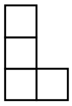
The figure below shows an L-shaped tromino.



1. How many ways are there to cut off an L-shaped tromino from a 6×6 grid?
2. How many ways are there to cut off an L-shaped tromino from a 7×9 grid?
3. How many ways are there to cut off an L-shaped tromino from a 8×8 grid?

Practice Problems III

The figure below shows an L-shaped tetromino.



1. How many ways are there to cut off an L-shaped tetromino from a 6×6 grid?
2. How many ways are there to cut off an L-shaped tetromino from a 7×9 grid?
3. How many ways are there to cut off an L-shaped tetromino from a 8×8 grid?

A Problem from a Real Math Competition

Today's problem comes from Canadian Mathematics Competition (CMC) – Grade 8 Gauss.

Problem

(CMC 2004 Grade 8 Gauss Problem 25)

A large block, which has dimensions n by 11 by 10, is made up of a number of unit cubes and one 2 by 1 by 1 block. There are exactly 2362 positions in which the 2 by 1 by 1 block can be placed. What is the value of n?

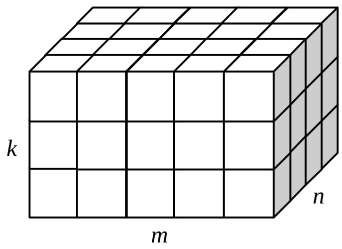
Answer: 8

Solution:

We will see how nicely "moving the shape" works in this problem.

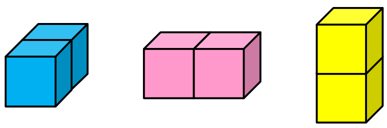
We write the problem in a different form:

The figure below shows a $k \times m \times n$ 3D grid.



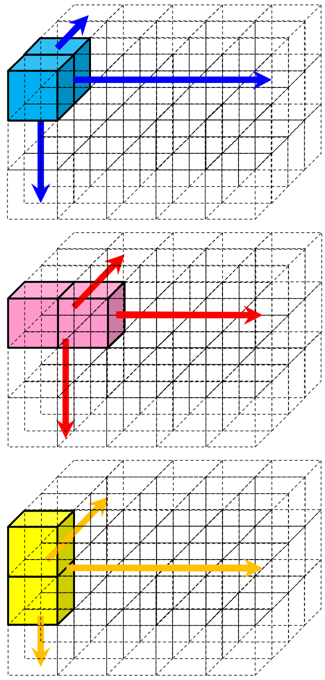
How many ways do you have to cut off a $1 \times 1 \times 2$ block along the grid lines from the $k \times m \times n$ grid?

The block has three different orientations:



Place each at one corner of the grid, move it, and then see how many different positions there are.

See the figures below. For the first orientation, there are $k \cdot m \cdot (n-1)$ positions to move the block. For the second, there are $k \cdot (m-1) \cdot n$ positions. For the third, there are $(k-1) \cdot m \cdot n$ positions.



Therefore, the number of ways to cut off the block is

$$k \cdot m \cdot (n-1) + k \cdot (m-1) \cdot n + (k-1) \cdot m \cdot n .$$

In the original problem, we are given $k = 11$ and $m = 10$. Plugging the numbers in we have

$$11 \cdot 10 \cdot (n-1) + 11 \cdot (10-1) \cdot n + (11-1) \cdot 10 \cdot n = 2362 .$$

Solving for n we obtain $n = 8$.

Practice Problem

A large block, which has dimensions n by 10 by 12, is made up of a number of unit cubes and one 1 by 2 by 3 block. There are exactly 7656 positions in which the 1 by 2 by 3 block can be placed. What is the value of n?

Answers to All Practice Problems in Last Issue

Last Number You Count

1. 109	2. 182	3. 81
4. 7184	5. 56,421	6. 2008
7. 5113	8. 2068	

How Do You Count: Identifying a Reference

Practice Problems I

1. 80
2. 164
3. $4(n-1)(n-2)$
4. $2(m-2) + 2(n-2) + 4(m-2)(n-2)$

Practice Problems II

1. 100
2. 192
3. 196

A Problem from a Real Math Competition

13,200

Solutions to Creative Thinking Problems 7 to 9

7. Odd Man

You may have your own pattern. But my pattern is:

First	→	1st	Second	→	2nd
Third	→	3rd	Fourth	→	4th

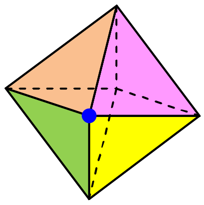
Therefore, **CD** is the odd man as the answer.

Is your pattern better than mine?

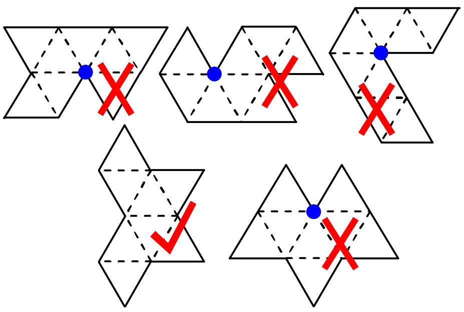
8. Display of an Octahedron

You may obtain the answer by using your imagination. But I have a nice solution.

Look at each of the six vertices of the octahedron. It touches four triangular faces. After the octahedron is displayed, any vertex touches at most four triangles.



But there is a vertex touching five triangles in each of the four figures crossed by ✗. The figure checked by ✓ is the answer.



9. Same Time and Same Place

Solution One:

Call the old man "Man *A*". Man *A* goes up the mountain on the first day, and down the mountain on the second.

Assume that another man called "Man *B*" goes down the mountain on the second day with the same paces as Man *A*. Now we have the following situation:

Man *A* goes up the mountain from 8:00 am to 8:00 pm on the first day, and Man *B* goes down the mountain from 8:00 am to 8:00 pm on the second day.

We overlap the two days into one day. That is, Man *A* goes up the mountain from 8:00 am to 8:00 pm on a day, and Man *B* goes down the mountain from 8:00 am to 8:00 pm on the same day along the same route.

What will happen?

Of course, Man *A* and Man *B* will meet.

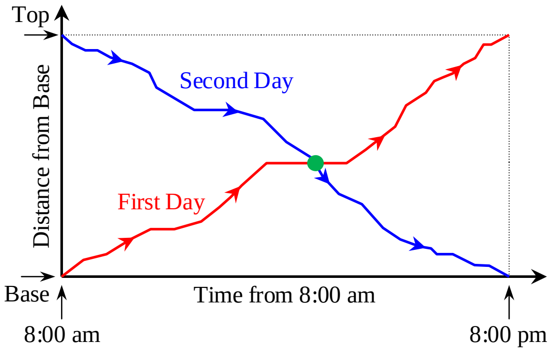
What does the meeting mean?

It means that at the same time Man *A* and Man *B* are at the same place.

Go back to the original problem. We can readily see that there was a time instance at which the old man was at the same place during the two days.

Solution Two:

Draw a diagram with time from 8:00 am as the x-axis and distance from the mountain base as the y-axis.



In the figure the red curve expresses the distance-time relationship for the old man on the first day, and the blue line on the second day. Both curves are continuous. The two curves must cross.

What does the crossing mean?

It means that at a time instance the old man was at the same place during the two days.

Creative Thinking Problems 10 to 12

10. What Comes Next?

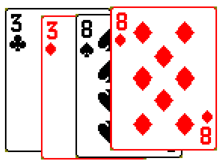
O T T F F S S ?

11. Get My Number

I have an integral number between 0 and 15 inclusive in my mind. Design four "yes or no" questions to be asked of me such that you will know my number after the completion of asking and answering.

12. Make 24 with 3, 3, 8, and 8

After you have the experience with 3, 3, 7, and 7, you may be able to make 24 with



Try it. See the rules in **Creative Thinking Problem 6** appearing in *Issue 2, Volume 1*.

(Solutions will be presented in the next issue.)