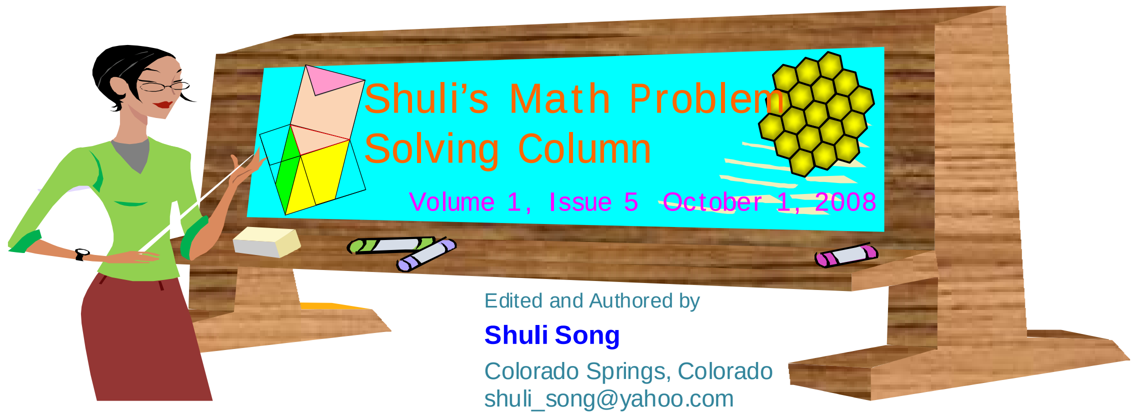

Edited and Authored by
Shuli Song
Colorado Springs, Colorado
shuli_song@yahoo.com

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Math Trick

Mental Calculation: $\overline{ba} \times \overline{ca}$

The Trick

This short lesson will present how to obtain the product for each of the following multiplications in 3 seconds:

$76 \times 36 =$	$42 \times 62 =$	$25 \times 85 =$
$84 \times 14 =$	$58 \times 58 =$	$49 \times 69 =$

There are two properties in common for all these pairs of two-digit numbers:

1. The sum of the two tens digits is 10.
2. The two ones digits are the same.

Write the multiplications in the general form: $\overline{ba} \times \overline{ca}$ where a, b, and c are digits with $b + c = 10$.

The two steps are shown through the following examples.

Example 1

Calculate 76×36.

Step 1: Calculate $b \times c + a$.

In this example, $7 \times 3 + 6 = 27$.

Step 2: Attach a^2 as a **TWO**-digit number to the right of the result in step 1.

In this example, attach $6^2 = 36$ to the right of 27: 2736

Now we are done: $76 \times 36 = 2736$.

Example 2

Calculate 42×62.

Step 1: Calculate $4 \times 6 + 2 = 26$

Step 2: Attach $2^2 = 4$, treated as two digits 04: 2604

Then $42 \times 62 = 2604$.

Why Does This Work?

Write $\overline{ba}$ and $\overline{ca}$ in the base 10 representation:

$$\overline{ba} = 10b + a \text{ and } \overline{ca} = 10c + a.$$

Then we have

$$\overline{ba} \times \overline{ca} = (10b + a)(10c + a) = 100bc + 10ab + 10ac + a^2$$
$$= 100bc + 10a(b + c) + bc.$$

Noting that $b + c = 10$ we obtain

$$\overline{ba} \times \overline{ca} = 100bc + 100a + a^2 = 100(bc + a) + a^2.$$

This shows that to calculate $\overline{ba} \times \overline{ca}$ we may calculate $bc + a$, multiply the result by 100, and add a^2.

That is, we may obtain the product by calculating $b \times c + a$ and attaching a^2 as a two-digit number to the right of $b \times c + a$.

Practice Problems

$48 \times 68 =$	$57 \times 57 =$	$35 \times 75 =$
$72 \times 32 =$	$19 \times 99 =$	$84 \times 24 =$
$96 \times 16 =$	$21 \times 81 =$	$54 \times 54 =$
$36 \times 76 =$	$88 \times 28 =$	$45 \times 65 =$
$18 \times 98 =$	$67 \times 47 =$	$89 \times 29 =$

Math Competition Skill

How Do You Count? – Separating Stamps

Problem

In the 2004 British Columbia College Senior High School Mathematics Contest (BCCSHSMC) there is a nice problem:

(BCCSHSMC 2004 Final Round Part A Problem 7)

Bob bought 12 postage stamps in a sheet of 3×4 stamps, as shown. Meg asks him to give her four stamps that are all joined together, with each stamp joined to the other three along at least one edge. Find the number of ways of separating four stamps.

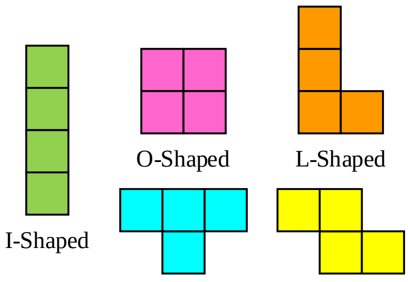
1	2	3	4
5	6	7	8
9	10	11	12

Solution

First we have to answer the following question:

What shapes can we have for a set of four connected stamps?

If we consider two shapes being the same when one can be overlapped with the other by rotating and/or flipping, then there are five different shapes of four connected stamps (tetrominos):



We will count the number of ways to have a set of four connected stamps in each of the five shapes.

We are equipped with tools from the last two issues to solve this problem. The "Moving the Shape" method works well.

For the I-shape, we can put it only horizontally, as shown in the figure below. We have three positions (including the present position) to move it vertically. We cannot move it horizontally.

5	6	7	8
9	10	11	12

So there are 3 ways to have a set of four connected stamps in the I-shape.

For the O-shape, we put it at the top-left corner. We have three positions (including the present position) to move it to the right and two positions to the bottom.

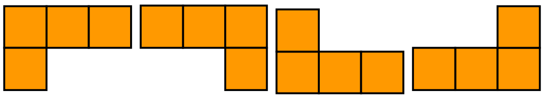
		3	4
		7	8
9	10	11	12

Thus there are $2 \times 3 = 6$ ways to have a set of four connected stamps in the O-shape.

For the L-shape, we can observe different orientations. There are two cases.

Case 1: Horizontal

There are four horizontal orientations:



For each of the four orientations, we place it at the top-left corner. Then we have two positions to move it to the right and two positions to the bottom.

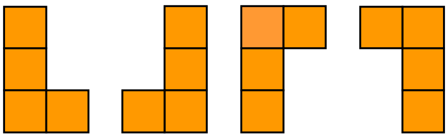
			4
	6	7	8
9	10	11	12

Hence there are $2 \times 2 = 4$ ways to have a set of four connected stamps in each orientation of the horizontal L-shape.

Altogether, there are $4 \times 4 = 16$ ways in the horizontal L-shape.

Case 2: Vertical

There are four vertical orientations as well:



For each of these orientations, we place it to the left side. We have three positions to move it horizontally. We cannot move it vertically.

	2	3	4
	6	7	8
		11	12

So there are 3 ways to have a set of four connected stamps in each orientation of the vertical L-shape.

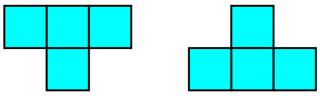
Altogether, there are $4 \times 3 = 12$ ways in the vertical L-shape.

Therefore, there are $16 + 12 = 28$ ways to have a set of four connected stamps in the L-shape.

For the T-shape, we also have different orientations. There are two cases.

Case 1: Horizontal

There are two horizontal orientations:



For each of the two orientations, we place it at the top-left corner. Then we have two positions to move it to the right and two positions to the bottom.

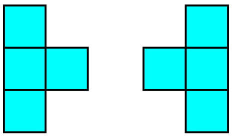
			4
5		7	8
9	10	11	12

There are $2 \times 2 = 4$ ways to have a set of four connected stamps in each orientation of the horizontal T-shape.

Altogether, there are $2 \times 4 = 8$ ways in the horizontal T-shape.

Case 2: Vertical

There also are two vertical orientations:



For each of these orientations, we put it to the left side. We have three positions to move it horizontally. We cannot move it vertically.

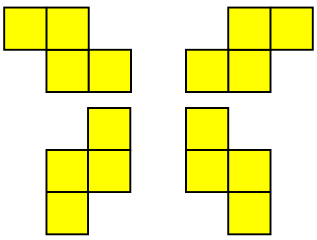
	2	3	4
		7	8
	10	11	12

So there are 3 ways to have a set of four connected stamps in each orientation of the vertical T-shape.

Altogether, there are $2 \times 3 = 6$ ways in the vertical T-shape.

Therefore, there are $8 + 6 = 14$ ways to have a set of four connected stamps in the T-shape.

For the Z-shape, there are two horizontal orientations and two vertical orientations:



There are also 8 ways in the horizontal Z-shape and 6 ways in the vertical Z-shape.

Altogether, there are $8 + 6 = 14$ ways in the Z-shape.

Finally, there are

$$3 + 6 + 28 + 14 + 14 = 65$$

ways to have a set of four connected stamps.

Practice Problems I

1. Andrew bought 15 postage stamps in a sheet of 3×5 stamps, as shown. Anna asks him to give her four stamps that are all joined together, with each stamp joined to the other three along at least one edge. Find the number of ways of separating four stamps.

1	2	3	4	5
6	7	8	9	10
11	12	13	14	15

2. Bob bought 20 postage stamps in a sheet of 4×5 stamps, as shown. Barbara asks him to give her four stamps that are all joined together, with each stamp joined to the other three along at least one edge. Find the number of ways of separating four stamps.

1	2	3	4	5
6	7	8	9	10
11	12	13	14	15
16	17	18	19	20

Practice Problems II

1. Al bought 12 postage stamps in a sheet of 3×4 stamps, as shown. Belle asks him to give her three stamps that are all joined together, with each stamp joined to the other two along at least one edge. Find the number of ways of separating three stamps.

1	2	3	4
5	6	7	8
9	10	11	12

2. Carl bought 15 postage stamps in a sheet of 3×5 stamps, as shown. Dianna asks him to give her three stamps that are all joined together, with each stamp joined to the other two along at least one edge. Find the number of ways of separating three stamps.

1	2	3	4	5
6	7	8	9	10
11	12	13	14	15

3. Jack bought 20 postage stamps in a sheet of 4×5 stamps, as shown. Jill asks him to give her three stamps that are all joined together, with each stamp joined to the other two along at least one edge. Find the number of ways of separating three stamps.

1	2	3	4	5
6	7	8	9	10
11	12	13	14	15
16	17	18	19	20

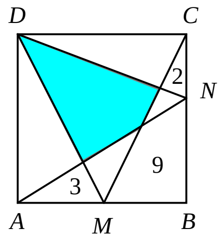
A Problem from a Real Math Competition

Today's problem comes from Math Kangaroo Contest.

Problem

(Math Kangaroo Contest 2006 Grades 9–10 Problem 28)

Points *M* and *N* are chosen on sides *AB* and *BC*, respectively, of square *ABCD*. The square is then divided into eight parts with three given areas. What is the area of the shaded region?

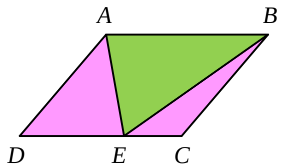


Answer: 14

Solution:

Let me introduce a tool:

ABCD is a parallelogram. *E* is a point on *CD*.

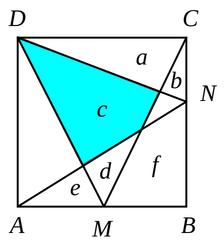


The area of ΔABE (shaded with green) is half the area of *ABCD*. The total area of ΔADE and ΔBCE (shaded with pink) is half the area of *ABCD* as well.

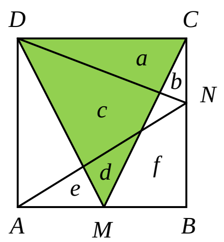
The proof is obvious.

With this tool the problem becomes easy. Note that a square is a special parallelogram.

Let *a*, *b*, *c*, *d*, *e*, and *f* represent the areas of the corresponding regions as shown below, where $b = 2$, $e = 3$, and $f = 9$. We have to find *c*.

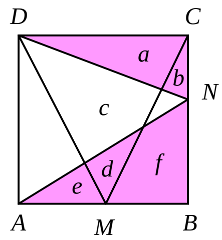


Look at the triangle shaded with green.



According to the tool, $a + c + d$ is equal to half the area of *ABCD*.

Look at the regions shaded with pink.



We see that $a + b + d + e + f$ is also equal to half the area of *ABCD*.

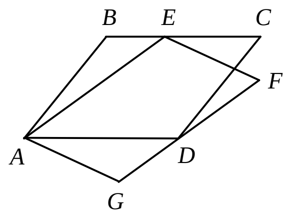
So $a + c + d = a + b + d + e + f$.

Therefore, $c = b + e + f = 2 + 3 + 9 = 14$.

Practice Problem

(Polya Mathematics Competition 2001 Individual Round Problem I4)

In the figure, *ABCD* and *AEFG* are parallelograms. *E* is on *BC* and *D* is on *FG* as shown. $AB = 6$, $AE = 7$, and $BE = 3$. What is the ratio of the area of *ABCD* to the area of *AEFG*?



Answers to All Practice Problems in Last Issue

Math Trick: Mental Calculation

Practice Problems I

7224	5625	3021
621	9009	2016
4209	216	2025
4221	7216	3024

Practice Problems II

11021	207016	990024
801009	245016	99900021

How Do You Count: Moving the Shape

Practice Problems I

1. 80
2. 164
3. $4(n-1)(n-2)$
4. $2(m-1)(n-2)+2(m-2)(n-1)$

Practice Problems II

1. 100
2. 192
3. 196

Practice Problems III

1. 160 2. 328 3. 336

A Problem from a Real Math Competition

14

Solutions to Creative Thinking Problems 10 to 12

10. What Comes Next?

People often list objects alphabetically. People also often list objects numerically. You may find the pattern by counting the natural numbers from 1.

My answer is **E** representing 8.

11. Get My Number

From how many numbers do I select a number?

It is $15-0+1=16$.

Note that $2^4=16$. If you can cut off a half of the numbers each time by asking one question, you will eventually get my number.

There are many ways to do this. You may have a way by finding a property which a half of the numbers possess, but the other half don't.

Here are several solutions.

Solution One:

Since there are 8 numbers smaller than 8, the first question may be:

Is the number smaller than 8?

If the answer is "yes", the second question may be:

Is the number smaller than 4?

If the answer is "no" to the first question, the second question may be:

Is the number smaller than 12?

Continuing this you will get my number.

Assume that my number is 10. We may have the following conversation:

Q: Is your number smaller than 8?

A: No.

Q: Is your number smaller than 12?

A: Yes.

Q: Is your number smaller than 10?

A: No.

Q: Is your number smaller than 11?

A: Yes.

Then the number is 10.

Solution Two:

A half of the 16 numbers are odd, and the other half are even. That is, a half of the numbers yield the same remainder upon division by 2. So you may ask the first question accordingly.

Then a half of the remaining 8 numbers have the same remainder upon division by 4. Thus the second question may come according to divisibility by 4.

Similarly, the third and fourth questions may be raised according to divisibility by 8 and 16, respectively.

Assume that my number is 11 this time. Then the conversation may be:

Q: Is the remainder 0 when the number is divided by 2?

A: No.

After this the remaining 8 numbers are 1, 3, 5, 7, 9, 11, 13, and 15, which yield remainders only 1 or 3 upon division by 4.

Q: Is the remainder 1 when the number is divided by 4?

A: No.

Then the remaining 4 numbers are 3, 7, 11, and 15, which yield remainders only 3 or 7 upon division by 8.

Q: Is the remainder 3 when the number is divided by 8?

A: Yes.

Now the remaining 2 numbers are 3 and 11, which yield remainders only 3 or 11 upon division by 16.

Q: Is the remainder 3 when the number is divided by 16?

A: No.

Then the number is 11.

In the first two solutions the second and following questions depend on the answers to the previous questions. The following solution is essentially the same as solutions one and two. However, it is nicer because the following questions don't depend on the answers to the previous questions.

Solution Three:

A number from 0 to 15 can be expressed in base 2 with four digits. For example, 13 in base 10 is 1101 in base 2.

Place 0's as the leading digits if the number of digits is smaller than 4. For example, 3 in base 10 is 11 in base 2, but we express it as 0011.

Because there are only two possibilities (0 or 1) in every digit position in binary numbers, we can ask a question to determine each of the digits.

Assume that 5 is my number.

Q: If your number is expressed in base 2, is the first (leftmost) digit 0?

A: Yes.

Q: If your number is expressed in base 2, is the second digit 0?

A: No.

Q: If your number is expressed in base 2, is the third digit 0?

A: Yes.

Q: If your number is expressed in base 2, is the forth digit 0?

A: No.

So the number is 0101 in base 2, which is equal to 5 in base 10.

Any ways to eliminate a half of the numbers each time so that we go from 16 numbers to one number yield a solution.

We may have solution four as follows.

Solution Four:

Assume that 15 is my number this time.

The first question may be:

Q: Is the number one of 0, 1, 3, 7, 8, 9, 10, and 15?

This question is not so elegant, but it works. With this question you can eliminate a half of the 16 numbers.

Note that there are 8 numbers in this question.

A: Yes.

Now we see four of the remaining eight numbers divisible by 3. Then the second question may be

Q: Is the number divisible by 3?

A: Yes.

Now four numbers remain: 0, 3, 9, and 15.

Note that 9 is a square number. If we count 0 as a square number ($0 = 0^2$) as well, then there are two square numbers.

So the third question may be:

Q: Is the number a square number?

A: No.

Now two numbers remain: 3 and 15.

The fourth question may simply be:

Q: Is the number 3?

A: No.

Or

Q: Is the number prime?

A: No.

Or

Q: Is the number divisible by 5?

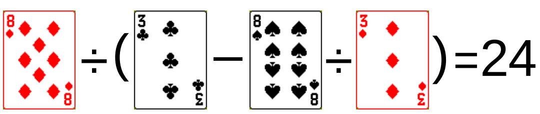
A: Yes.

Therefore, the number is 15.

12. Make 24 with 3, 3, 8, and 8

We may often think of $8 \times 3 = 24$ in playing this game. We may not think of $8 \div \frac{1}{3} = 24$. Using 3, 3, and 8 to make $\frac{1}{3}$ leads to the solution.

My answer is:



Creative Thinking Problems 13 to 15

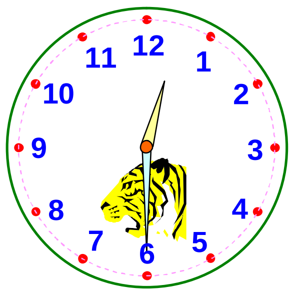
13. Ten Coins

Ten coins are arranged in four rows as shown below.

At least how many coins do you have to move such that the coins are arranged as follows:

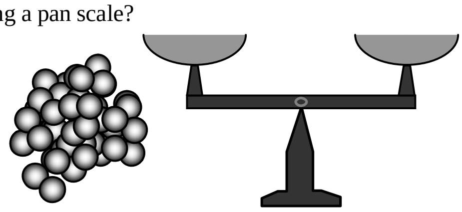
14. Broken Clock

A traditional clock fell down and broke into three pieces. Each piece happened to have four numbers, and the sum of the numbers on each piece was the same. How did the clock break?



15. 81 Balls

There are 81 balls that look exactly the same. Three out of them are bad. The three bad balls have the same weight and are slightly lighter than the good balls. All good balls have the same weight. At least how many times of weighing do you need to find 20 good balls by using a pan scale?



(Solutions will be presented in the next issue.)

