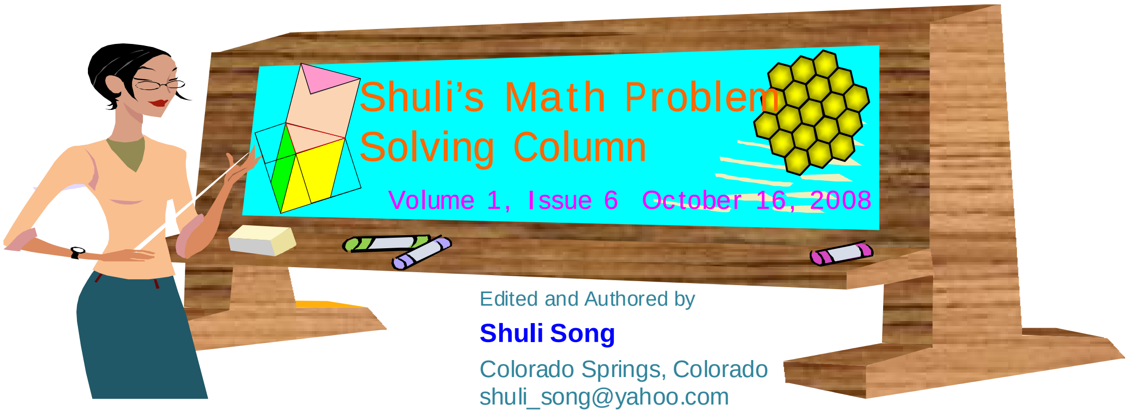


Edited and Authored by
Shuli Song
Colorado Springs, Colorado
shuli_song@yahoo.com

Content

Math Trick

Mental Calculation: $(\overline{a5})^2$

The Trick

Can you calculate the following squares mentally?

$35^2 =$	$75^2 =$	$125^2 =$
$55^2 =$	$85^2 =$	$455^2 =$

Recall! You have learned this skill in *Issue 4, Volume 1*.

In fact, they are in the form $\overline{ab} \times \overline{ac}$ with $b + c = 10$.

If a number ends in 5, we have a short cut to calculate the square of the number. The steps are shown through the following examples.

Example 1

Calculate 45^2.

Step 1: Calculate $a \times (a+1)$. In this example, $4 \times 5 = 20$.

Step 2: Attach 25 to the right of the result in step 1. In this example, attach 25 to the right of 20.

Now we are done: $45^2 = 2025$.

Example 2

Calculate 75^2.

Step 1: $7 \times 8 = 56$

Step 2: Attach 25 to the right of 56.

Then $75^2 = 5625$.

It works for a number with three or more digits as well.

Example 3

Calculate 345^2.

Step 1: $34 \times 35 = 1190$. To calculate 34×35 we may use the trick again:

$34 \times 35 = 35^2 - 35 = 1225 - 35 = 1190$.

Step 2: Attach 25.

We have $345^2 = 119025$.

Why Does This Work?

Write $\overline{a5}$ in the base 10 representation:

$$\overline{a5} = 10a + 5.$$

Then we have

$$(\overline{a5})^2 = (10a+5)^2 = 100a^2 + 100a + 25$$
$$= 100a(a+1) + 25.$$

This shows that to calculate $(\overline{a5})^2$ we may calculate $a(a+1)$, multiply the result by 100, and add 25.

That is, we may obtain the answer by calculating $a(a+1)$ and attaching 25 to the right of $a(a+1)$.

Practice Problems I

$25^2 =$	$95^2 =$	$65^2 =$
$55^2 =$	$15^2 =$	$85^2 =$
$35^2 =$	$75^2 =$	$45^2 =$

Practice Problems II

$105^2 =$	$195^2 =$	$755^2 =$
$245^2 =$	$555^2 =$	$395^2 =$
$895^2 =$	$645^2 =$	$1005^2 =$

Math Competition Skill

Divisibility by 2, 4, 8, etc.

Introduction Problem

Let me introduce a nice problem from International Mathematical Talent Search (IMTS)

(IMTS 1999 Round 32 Problem 1)

Exhibit a 13-digit integer N that is an integer multiple of 2^{13} and whose digits consist of only 8s and 9s.

The problem asks for a number that is divisible by 2^{13}. If a number is divisible by 2^{13}, it is obviously divisible by 2^n for any positive integer $n \le 13$. That is, it is divisible by 2, 4, 8, etc.

To solve the problem we need the basic knowledge about divisibility by $2^1 = 2$, $2^2 = 4$, $2^3 = 8$, $\cdots$, 2^n, $\cdots$.

Divisibility by 2^n

Let N be a positive integer.

We know how to determine whether N is divisible by 2. A textbook often teaches that if N ends in 0, 2, 4, 6, or 8, then N is divisible by 2, and vice versa.

We put the statement in a different way:

If the ones digit of N is divisible by 2 (the ones digit must be 0, 2, 4, 6, or 8), then N is divisible by 2, and vice versa.

How do we see whether N is divisible by 4?

The following theorem will bring you a short cut:

If the number formed by the last two digits (tens and ones) of N is divisible by 4, then N is divisible by 4, and vice versa.

For example, is 86420 divisible by 4?

Since 20 is divisible by 4, 86420 is divisible by 4.

How do we see whether N is divisible by 8?

We have a similar theorem:

If the number formed by the last three digits (hundreds, tens, and ones) of N is divisible by 8, then N is divisible by 8, and vice versa.

Is 86420 divisible by 8?

Since 420 is not divisible by 8, 86420 is not divisible by 8.

In general we have the following theorem:

Let n be a positive integer. If the number formed by the last n digits of N is divisible by 2^n, then N is divisible by 2^n, and vice versa.

As an example, is 4395061728 divisible by 32?

Note that $32 = 2^5$. Since 61728 is divisible by 32, 4395061728 is divisible by 32.

Proof of the General Theorem

Let N be an m-digit number where $m \ge n$. Let $a_1, a_2, \cdots, a_m$ be the m digits indexed starting from the ones digit. That is,

$$N = \overline{a_m \cdots a_2 a_1}.$$

Note that

$$N = \overline{a_m \cdots a_2 a_1} = \overline{a_m \cdots a_{n+2} a_{n+1}} \cdot 10^n + \overline{a_n \cdots a_2 a_1}.$$

Since $10^n = 2^n \cdot 5^n$ is divisible by 2^n, we just need to look at the second part.

Therefore, if $\overline{a_n \cdots a_2 a_1}$ is divisible by 2^n, then $N = \overline{a_m \cdots a_2 a_1}$ is divisible by 2^n, and vice versa.

Solution to the Introduction Problem

Let us build the 13-digit number gradually from the ones digit.

The number must be divisible by 2. So the ones digit is 8.

The number must be divisible by 4. So we should test weather 88 or 98 is divisible by 4. It is obvious that the tens digit is 8.

The number must be divisible by 8. Note that 888 is divisible by 8, and 988 is not. So the hundreds digit is 8.

Furthermore, the number must be divisible by 16. By testing, 8888 is not divisible by 16. So 9888 is the last four digits of the number.

Continue with this process. The 13-digit number is

8,898,989,989,888.

A Problem from AIME

(American Invitational Mathematics Examination 2003 – Alternative Problem 2)

Let N be the greatest integer multiple of 8, no two of whose digits are the same. What is the remainder when N is divided by 1000?

Answer: 120

Solution:

To make the greatest number we must use all 10 different digits. Of course, we will use each digit once.

9876543210 is the greatest number, no two of whose digits are the same. But it is not divisible by 8 since 210 is not divisible by 8.

Now we have to adjust the last three digits. By simple calculations, 120 is divisible by 8. So N = 9876543120.

The answer to the problem is 120.

Practice Problems I

1. Circle the numbers divisible by 4:

   1992 2000 3765 188 24680 1768 2004 9090 280642

2. Circle the numbers divisible by 8:
   5000 12008 56222 12345 24680 87456 77444 24016
3. Circle the numbers divisible by 16:
   90000 2008 69136 54321 62480 23456 888888 12624

Practice Problems II

1. Find the 10-digit number which is a multiple of 1024 and consists of only 1s and 2s as its digits.
2. Let N be the smallest 10-digit integer which is a multiple of 8, no two of whose digits are the same. What is the remainder when N is divided by 1000?

A Problem from a Real Math Competition

Today's problem comes from MathCounts.

Problem

(MathCounts 1991 National Sprint Problem 28)

A man is running through a train tunnel. When he is $2/3$ of the way through, he hears a train that is approaching the tunnel from behind him at a speed of 60 mph. Weather he runs ahead or back, he will reach an end of the tunnel at the same time the train reaches that end. At what rate, in miles per hour, is he running? (Assume he runs at a constant rate.)

Answer: 20

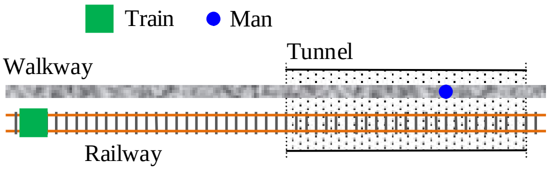
Solution:

This is a nice time-speed-distance problem. The key observation is that the traveling time is the same for the man and the train. It is true for both cases:

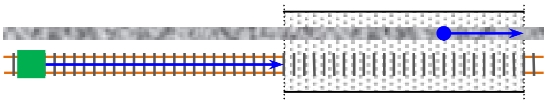
Case 1: The man travels ahead.

Case 2: The man travels back.

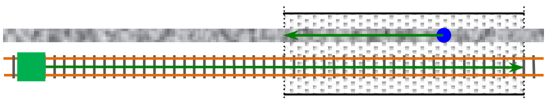
Look at the figures:



Positions when the man hears the train



Traveling distances by the man and the train in case 1



Traveling distances by the man and the train in case 2

In case 1 the man travels $1/3$ of the tunnel length, and the train travels the distance outside the tunnel.

In case 2 the man travels $2/3$ of the tunnel length, and the train travels the distance outside the tunnel plus the length of the whole tunnel.

The distance difference of the man between the two cases is $1/3$ of the tunnel length, while the distance difference of the train is the tunnel length.

Because the man and the train travel in the same time in the two cases, the distance ratio equals the speed ratio.

Let x be the speed of the man in miles per hour. Then

$$\frac{1/3}{1} = \frac{x}{60}$$

Solving this we obtain $x = 20$ as the answer.

Practice Problem

(Canadian Math Competition 2001 Grade 8 Gauss Problem 25)

Tony and Maria are training for a race by running all the way up and down a 700 m long ski slope. They each run up the slope at different constant speeds. Coming down the slope, each runs at double his or her uphill speed. Maria reaches the top first, and immediately starts running back down, meeting Tony 70 m from the top. When Maria reaches the bottom, how far behind is Tony?

Answers to All Practice Problems in Last Issue

Math Trick: Mental Calculation

3264	3249	2625
2304	1881	2016
1536	1701	2916
2736	2464	2925
1764	3149	2581

Separating Stamps

Practice Problems I

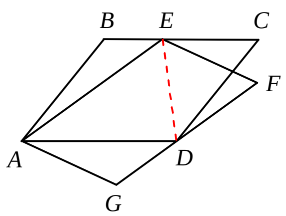
1. 94 2. 161

Practice Problems II

1. 34 2. 46 3. 70

A Problem from a Real Math Competition

The answer is 1. Draw *DE*. You will see the solution. All numbers in the problem are unnecessary.

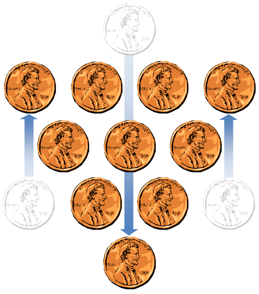


Solutions to Creative Thinking Problems 13 to 15

13. Ten Coins

I overlap the two sets of 10 coins:

At most 7 individual coins can be overlapped. Therefore, we have to move at least three coins. Here is the solution by moving three coins:



14. Broken Clock

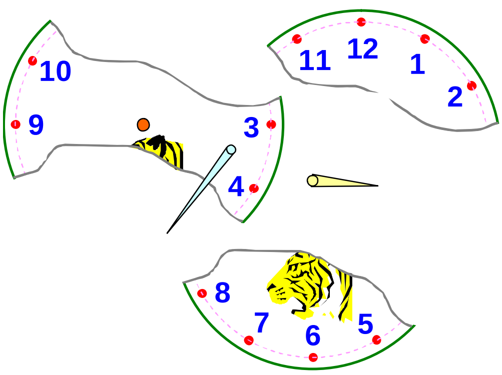
What is the sum of 12 numbers?

$$1+2+3+4+5+6+7+8+9+10+11+12=78.$$

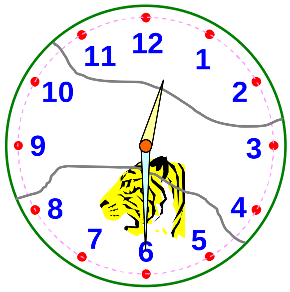
The clock broke into three pieces with each having four numbers of the same sum. This sum is $\frac{78}{3}=26$.

Now it is easy to know how the clock broke. We just need to divide the 12 numbers into 3 equal groups, each of which has the sum of 26. The easiest way may be to pair the numbers: 1 with 12, 2 with 11, 3 with 10, etc.

These are my broken pieces:



After I have assembled these pieces, the clock looks like:



15. 81 Balls

Place 40 balls onto each pan with one ball put aside. There are two cases for the scale:

Case a: In balance

Because there are three bad balls, the ball put aside is definitely bad, and each pan contains one bad ball. Divide any group of 40 balls into two groups of 20 balls. Place these two groups onto the pans. The scale is surely not in balance. The 20 balls in the heavier side are all good.

Case b: Not in balance

The heavier side contains at most one bad ball. Divide the 40 balls in the heavier side into two groups of 20 balls, and place these two groups onto the two pans. If the scale is in balance, the 20 balls on any pan are all good. If the scale is not in balance, the 20 balls in the heavier side are all good.

I have to weigh twice to find 20 good balls.

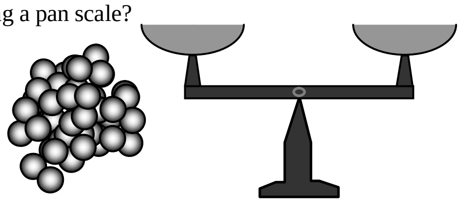
Creative Thinking Problems 16 to 18

16. $2+3\neq 5$

When is $2+3$ not equal to 5?

17. Another 81 Balls

There are 81 balls that look exactly the same. One out of the 81 balls is bad. All good balls have the same weight, but the bad ball is slightly lighter. At least how many times of weighing do you need to identify the bad ball by using a pan scale?



18. A Popular Pattern

This pattern is quite popular. What comes next?

1, 11, 21, 1112, 3112, 211213, 312213, 212223, …

(Solutions will be presented in the next issue.)

