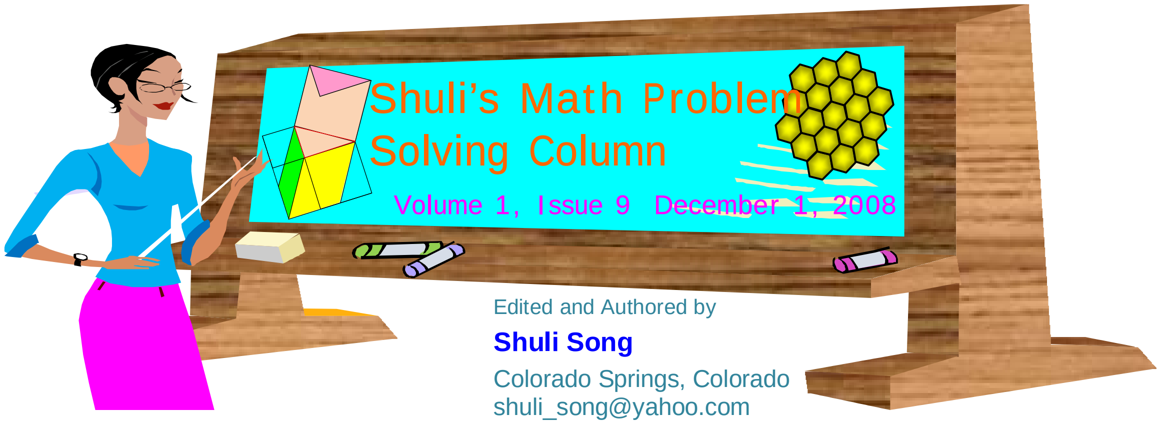


Content

Math Trick

Mental Calculation: $\overline{a1} \times \overline{b1}$

The Trick

Mentally calculate:

$71 \times 21 =$ $\quad 61 \times 51 =$ $\quad 31 \times 81 =$

$61^2 =$ $\quad 91 \times 41 =$ $\quad 41 \times 71 =$

Write these multiplications in the general form: $\overline{a1} \times \overline{b1}$ where *a* and *b* are digits.

The short cut to calculate the multiplications is shown through the following examples.

Example 1

Calculate 61×71.

Step 1: Calculate $a \times b$. In this example, $6 \times 7 = 42$.

Step 2: Calculate $a + b$. In this example, $6 + 7 = 13$.

Step 3: "Add" them this way:

$$\begin{array}{rccc} & 4 & 2 & \\ + & & 1 & 3 \\ \hline & 4 & 3 & 3 \end{array}$$

Step 4: Attach 1 to the right of the result in step 3: 4331.

We are done: $61 \times 71 = 4331$.

Example 2

Calculate 51×31.

Step 1: Calculate $5 \times 3 = 15$.

Step 2: Calculate $5 + 3 = 8$.

Step 3: "Add" them:

$$\begin{array}{rccc} & 1 & 5 & \\ + & & & 8 \\ \hline & 1 & 5 & 8 \end{array}$$

Step 4: Attach 1: 1581.

We have $51 \times 31 = 1581$.

It works for numbers with three or more digits.

Example 3

Calculate 251×61.

Step 1: Calculate $25 \times 6 = 150$.

Step 2: Calculate $25 + 6 = 31$.

Step 3: "Add" them:

$$\begin{array}{rcccc} & 1 & 5 & 0 & \\ + & & & 3 & 1 \\ \hline & 1 & 5 & 3 & 1 \end{array}$$

Step 4: Attach 1: 15311.

We obtain $251 \times 61 = 15311$.

Two more complicated examples are presented using the mental calculation skills presented in last issues.

Example 4

Calculate 371×771.

Step 1: Calculate $37 \times 77 = 2849$.

Recall how to calculate 37×77 mentally. If you have forgotten, go to *Issue 5, Volume 1*.

Step 2: Calculate $37 + 77 = 114$.

Step 3: "Add" them:

$$\begin{array}{rccccc} & 2 & 8 & 4 & 9 & \\ + & & & 1 & 1 & 4 \\ \hline & 2 & 8 & 6 & 0 & 4 \end{array}$$

Step 4: Attach 1: 286041.

Then $371 \times 771 = 286041$.

Example 5

Calculate 7451^2.

Step 1: Calculate 745^2.

First calculate

$$74 \times 75 = 75^2 - 75 = 5625 - 75 = 5550.$$

So $745^2 = 555025$.

Step 2: Calculate $745 + 745 = 1490$.

Step 3: "Add" them:

	5	5	5	0	2	5	
+				1	4	9	0
	5	5	5	1	7	4	0

Step 4: Attach 1: 55517401.

We have $7451^2 = 55517401$.

Why Does This Work?

Write $\overline{a1}$ and $\overline{b1}$ in the base 10 representation:

$$\overline{a1} = 10a + 1 \text{ and } \overline{b1} = 10b + 1.$$

So we have

$$\overline{a1} \times \overline{b1} = (10a+1)\times(10b+1)$$
$$= 100ab + 10a + 10b + 1 = 10 \times [10ab + (a+b)] + 1.$$

This shows that to calculate $\overline{a1} \times \overline{b1}$, we may do

Step 1: Calculate $a \times b$.

Step 2: Calculate $a + b$.

Step 3: "Add" them this way: $10ab + (a+b)$.

Step 4: Attach 1 to the right of the result in step 3.

Practice Problems I

$21 \times 91 =$	$31 \times 71 =$	$81 \times 41 =$
$71 \times 61 =$	$21 \times 51 =$	$61 \times 91 =$
$11 \times 91 =$	$81 \times 31 =$	$91 \times 81 =$
$31 \times 41 =$	$61 \times 51 =$	$71 \times 21 =$
$81^2 =$	$41^2 =$	$31^2 =$
$51^2 =$	$71^2 =$	$91^2 =$

Practice Problems II

$241 \times 31 =$	$41 \times 451 =$	$991 \times 91 =$
$751 \times 61 =$	$71 \times 801 =$	$61 \times 911 =$
$111 \times 341 =$	$251 \times 441 =$	$121 \times 131 =$
$721 \times 781 =$	$491 \times 691 =$	$751 \times 741 =$
$171 \times 191 =$	$611 \times 411 =$	$451^2 =$

Practice Problems III

$111 \times 12341 =$	$1251 \times 81 =$	$301 \times 3331 =$
$2551^2 =$	$3451^2 =$	$8911 \times 8991 =$

Math Competition Skill

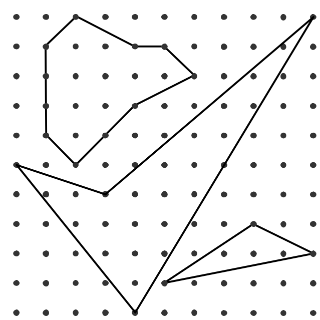
Pick's Theorem

Introduction

In mathematical contests at elementary and junior high levels, finding the area of a lattice polygon is a popular problem.

First what is a lattice polygon?

Points in the plane with integral coordinates are called *Lattice Points*. A polygon is called a *Lattice Polygon* if all vertices of the polygon are lattice points. The following figure shows three lattice polygons.



This short lesson will introduce Pick's Theorem, which can be used to calculate the area of a lattice polygon.

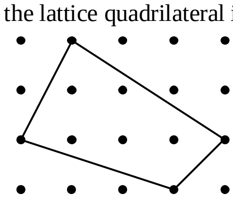
In the context all points are spaced one unit apart, horizontally and vertically if it is not specified otherwise.

A Problem with Three Solutions

Let us proceed with a problem.

Example 1

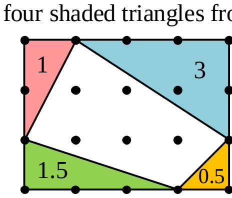
Find the area of the lattice quadrilateral in the figure.



Answer: 6

Solution One:

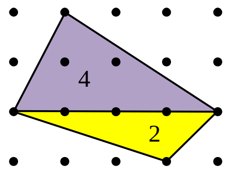
Calculate the area of the rectangle shown. Then subtract the areas of the four shaded triangles from it.



The area of the rectangle is $3 \times 4 = 12$. The areas of the four shaded triangles are marked in the figure. So the area of the quadrilateral is $12 - (1 + 3 + 0.5 + 1.5) = 6$.

Solution Two:

We can divide the quadrilateral into two triangles whose areas are 4 and 2 respectively.



The area of the quadrilateral is $4+2=6$.

The third solution will be given after we introduce Pick's Theorem.

Pick's Theorem

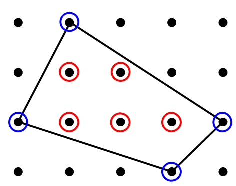
Let i be the number of lattice points in the interior of a lattice polygon, b be the number of lattice points on its boundary. Then the area of the polygon is:

$$A=i+\frac{b}{2}-1.$$

This is called *Pick's Theorem*.

Solution Three to Example 1:

In this problem there are five lattice points inside the quadrilateral, which are circled in red in the figure below. So $i=5$. There are four points on the boundary, which are circled in blue. Then $b=4$.



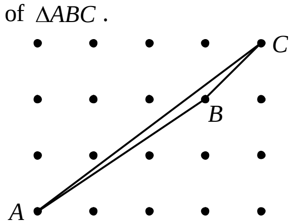
The area of the quadrilateral is $5+\frac{4}{2}-1=6$.

More Examples

Example 2

(12th AMC8 1996 Problem 22)

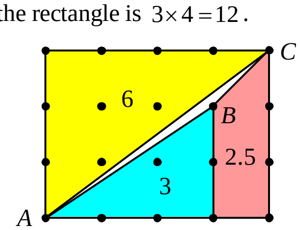
Find the area of ΔABC.



Answer: 0.5

Solution One:

The area of the rectangle is $3\times 4=12$.



The areas of the two shaded triangles and the shaded trapezoid are marked in the figure.

The area of ΔABC is $12-(6+3+2.5)=0.5$.

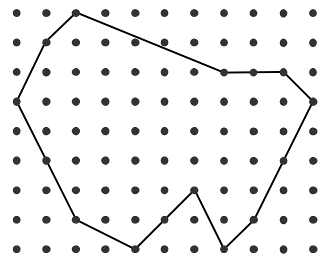
Solution Two:

Pick's Theorem is especially useful for this problem.

There are no lattice points inside the triangle, and there are 3 lattice points on its boundary. So the area of the triangle is $0+\frac{3}{2}-1=0.5$.

Example 3

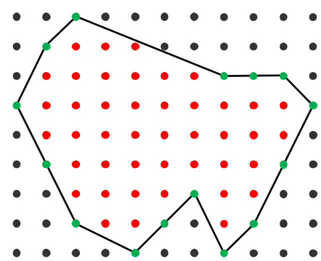
Find the area of the lattice polygon.



Answer: 49.5

Solution:

You may solve it using a different way. However, counting lattice points is easier for this problem. I color the interior points in red and the points on its boundary in green.

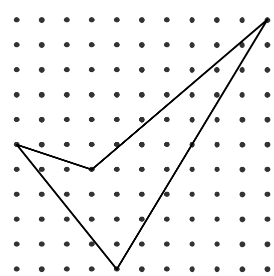


There are 43 red points and 15 green points. So the area of the polygon is $43+\frac{15}{2}-1=49.5$.

Example 4

(20th AMC8 2004 Problem 22)

What is the area enclosed by the geoboard quadrilateral below?

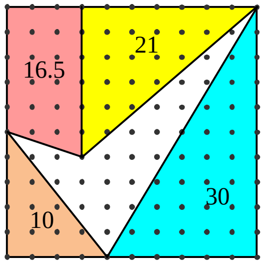


Answer: 22.5

Solution One:

The area of the square is $10\times 10=100$.

The areas of the four shaded regions is 16.5, 21, 30, and 10 respectively.

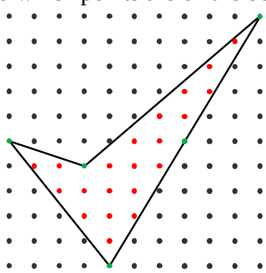


So the area of the quadrilateral is

$$100-(16.5+21+10+30)=22.5\,.$$

Solution Two:

Now we count the points. In this problem we have to be careful to identify which points are inside, which points are outside, and which points are on the boundary.



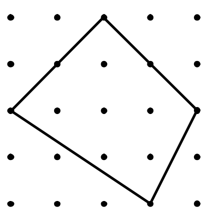
Again, I color the interior points in red and the points on the boundary in green.

There are 21 red points and 5 green points. So the area of the quadrilateral is

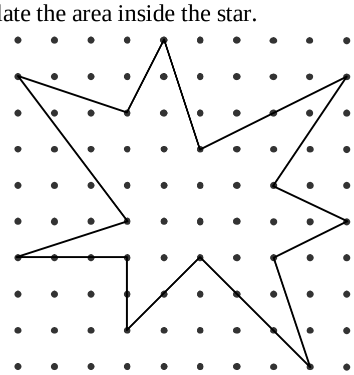
$$21+\frac{5}{2}-1=22.5\,.$$

Practice Problems I

1. Calculate the area of the quadrilateral in the figure using the three methods demonstrated in example 1.



2. Calculate the area inside the star.

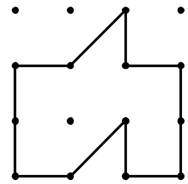


Practice Problems II

All problems in this section are from American Mathematics Competition Grade 8 (AMC8).

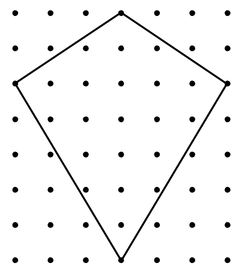
1. (14th AMC8 1998 Problem 6)

Dots are spaced one unit apart, horizontally and vertically. Find the number of square units enclosed by the polygon.



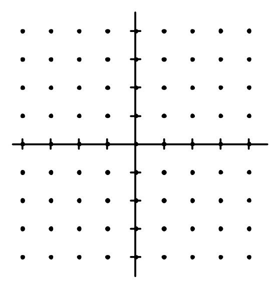
2. (17th AMC8 2001 Problem 7)

What is the area of the kite?



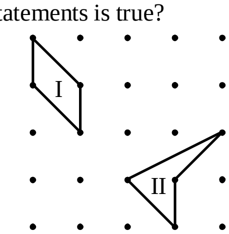
3. (17th AMC8 2001 Problem 11)

Points A, B, C and D have these coordinates: $A(3,2)$, $B(3,-2)$, $C(-3,-2)$ and $D(-3,0)$. Find the area of quadrilateral $ABCD$.



4. (16th AMC8 2000 Problem 18)

Consider these two geoboard quadrilaterals. Which of the following statements is true?



A) The area of quadrilateral I is more than the area of quadrilateral II.

B) The area of quadrilateral I is less than the area of quadrilateral II.

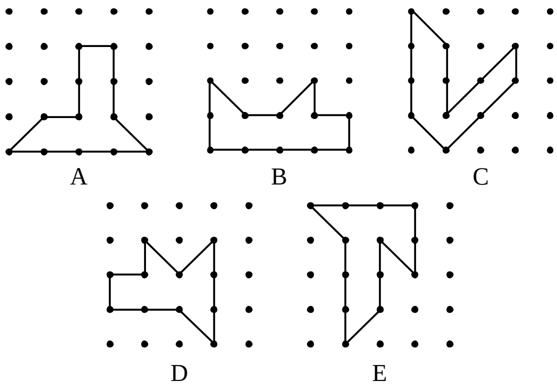
C) The quadrilaterals have the same area and the same perimeter.

D) The quadrilaterals have the same area, but the perimeter of I is more than the perimeter of II.

E) The quadrilaterals have the same area, but the perimeter of I is less than the perimeter of II.

5. (18th AMC8 2002 Problem 15)

Which of the following polygons has the largest area?



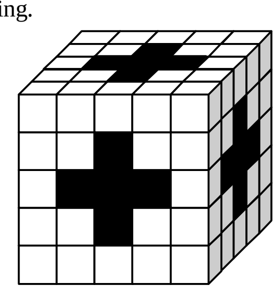
A Problem from a Real Math Competition

Today's problem comes from Canadian Mathematics Competition (CMC).

Problem

(CMC 1990 Grade 8 Gauss Problem 25)

A 5 by 5 by 5 cube is formed using 1 by 1 by 1 cubes. A number of the smaller cubes are removed by punching out the 15 designated columns from front to back, top to bottom, and right to left. Find the number of smaller cubes remaining.

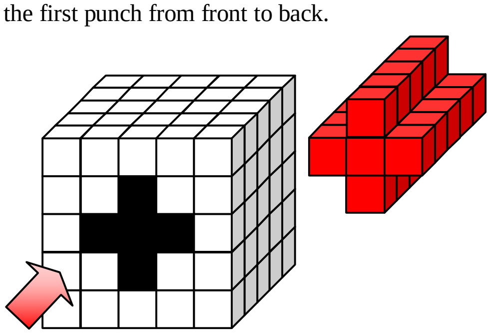


Answer: 76

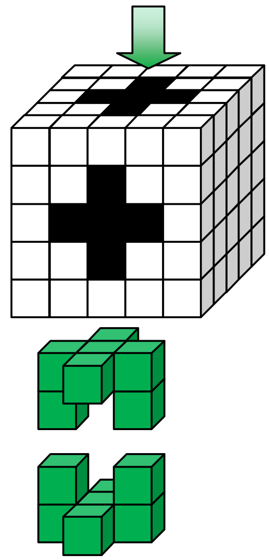
Solution One:

There are $5 \times 5 \times 5 = 125$ unit cubes before any punch.

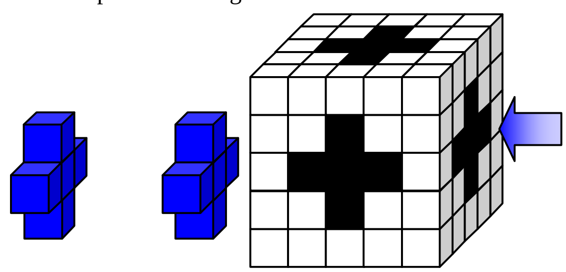
As shown below, we remove $5 \times 5 = 25$ unit cubes with the first punch from front to back.



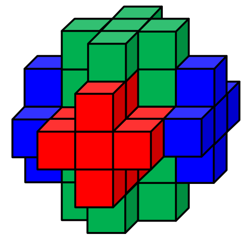
We remove another $5+2+0+2+5=14$ unit cubes with the second punch from top to bottom:



We remove $5+0+0+0+5=10$ unit cubes more with the third punch from right to left:



Altogether, we remove $25+14+10=49$ cubes. All the 49 removed unit cubes are assembled to form the solid shown below.

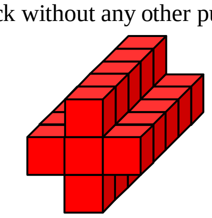


Therefore, the number of unit cubes remaining is

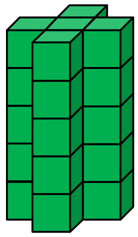
$$125-49=76.$$

Solution Two:

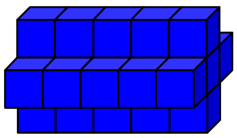
Let A be the set of unit cubes (red) removed by punching from front to back without any other punches.



Let B be the set of unit cubes (green) removed by punching from top to bottom without any other punches.



Let C be the set of unit cubes (blue) removed by punching from right to left without any other punches.



Let S be the set of unit cubes removed altogether by the three punches.

Use $|X|$ to denote the number of elements in set X.

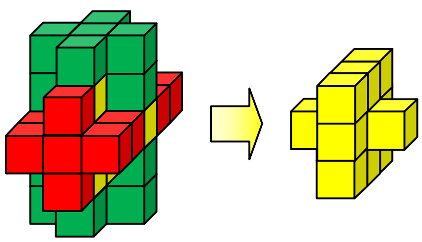
By the Inclusive-Exclusive Principle we have

$$|S| = |A| + |B| + |C| - |A \cap B| - |B \cap C| - |C \cap A| + |A \cap B \cap C| .$$

We know that $|A| = |B| = |C| = 25$.

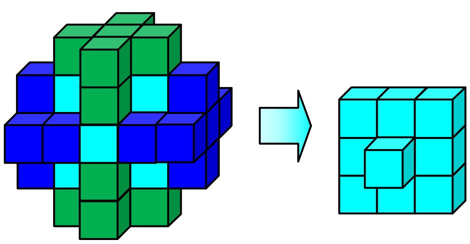
What is $|A \cap B|$?

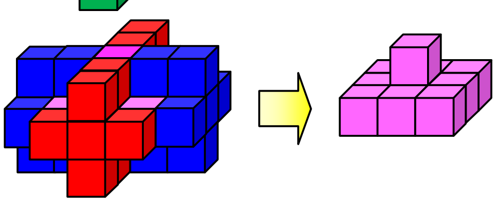
There are 11 common unit cubes (yellow) in sets A and B, as shown below.



So $|A \cap B| = 11$.

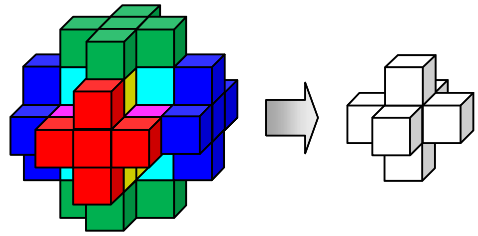
In the figures below, the unit cubes in $B \cap C$ are colored in cyan, and the unit cubes in $C \cap A$ are colored in pink. We have $|B \cap C| = 11$ and $|C \cap A| = 11$.





At last, what is $|A \cap B \cap C|$?

There are 7 common unit cubes in sets A, B, and C, as shown below. That is, $|A \cap B \cap C| = 7$.



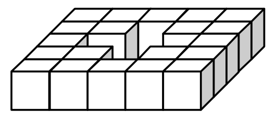
So the number of unit cubes removed is

$$|S| = 25 + 25 + 25 - 11 - 11 - 11 + 7 = 49 .$$

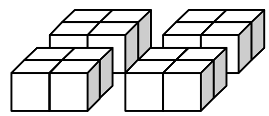
The number of unit cubes remaining is $125 - 49 = 76$.

Solution Three:

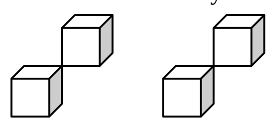
There are five layers of unit cubes counting from top to bottom. There are 20 unit cubes in the first and fifth layers:



There are 16 unit cubes in the second and fourth layers:



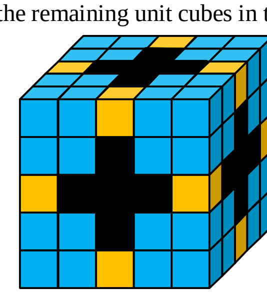
There are 4 unit cubes in the third layer:



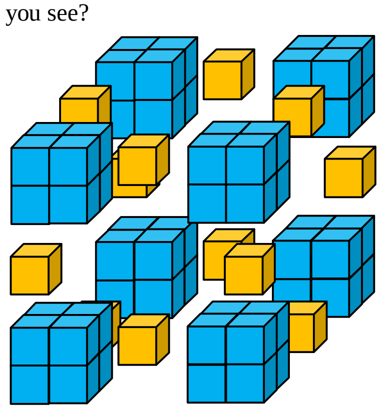
Altogether, there are $2 \times 20 + 2 \times 16 + 4 = 76$ unit cubes.

Solution Four:

I color all of the remaining unit cubes in two colors:



What do you see?

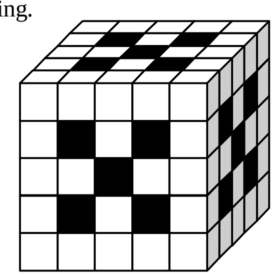


We see eight $2 \times 2 \times 2$ cubes (blue) at the corners of the original solid and 12 unit cubes (orange) at the edges. Therefore, there are $8 \times 8 + 12 \times 1 = 76$ unit cubes.

Practice Problem

(CMC 1990 Grade 7 Gauss Problem 25)

A 5 by 5 by 5 cube is formed using 1 by 1 cubes. A number of the smaller cubes are removed by punching out the 15 designated columns from front to back, top to bottom, and right to left. Find the number of smaller cubes remaining.



Answers to All Practice Problems in Last Issue

Math Trick: Mental Calculation

228	221	252
272	180	304
209	234	342
182	240	204
121	196	169
225	289	361
324	144	256

Chickens and Rabbits

Practice Problems I

1. 7 chickens and 4 rabbits
2. 17 chickens and 15 rabbits
3. 35 chickens and 65 rabbits

Practice Problems II

1. 8 boys and 6 boys
2. 11 people in Group *A* and 14 in Group *B*
3. 7 nickels
4. 25 seniors and 75 juniors
5. 5 tricycles

Practice Problems III

1. 15 nickels
2. 8 stamps of 86¢
3. 20 children
4. 4 seniors and 60 juniors

Practice Problems IV

1. 10 spiders, 20 dragonflies, and 30 cicadas
2. 3 spiders, 9 dragonflies, and 27 cicadas

A Problem from a Real Math Competition

99

Clues to Creative Thinking Problems 22 to 24

22. Route of 100

You may find a solution by trial and error. However, considering one of the two extremal routes below, you can find a solution easily.

Route with the maximum sum:

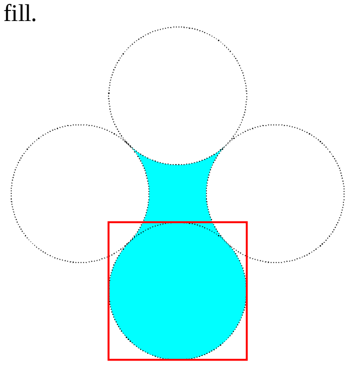
1↓	1	1	1	1	1	1	1	1
2↓	2	2	2	2	2	2	2	2
3↓	3	3	3	3	3	3	3	3
4↓	4	4	4	4	4	4	4	4
5↓	5	5	5	5	5	5	5	5
6↓	6	6	6	6	6	6	6	6
7↓	7	7	7	7	7	7	7	7
8↓	8	8	8	8	8	8	8	8
9→	9→	9→	9→	9→	9→	9→	9→	9

Route with the minimum sum:

1→	1→	1→	1→	1→	1→	1→	1→	1↓
2	2	2	2	2	2	2	2	2↓
3	3	3	3	3	3	3	3	3↓
4	4	4	4	4	4	4	4	4↓
5	5	5	5	5	5	5	5	5↓
6	6	6	6	6	6	6	6	6↓
7	7	7	7	7	7	7	7	7↓
8	8	8	8	8	8	8	8	8↓
9	9	9	9	9	9	9	9	9

23. Cutting the Vase

How do we divide the vase? It depends on how we would draw the square. For example, if we draw a square as shown below, we have to cut the vase into at least five pieces to make the square because we have four blank corners to fill.

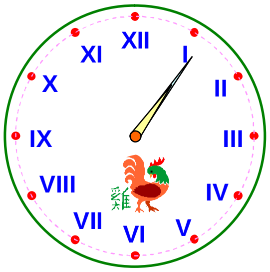


Now try to find a solution by yourself.

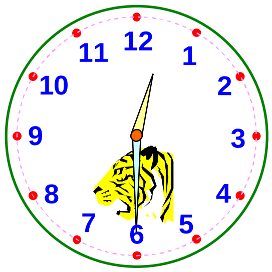
24. Broken Clock Again

We had an experience with the first "Broken Clock" problem. We may say that it is impossible in this problem because $1+2+\cdots+12=78$ is not divisible by 4.

But this clock is different from that clock. Compare them:



This Clock



That Clock

What difference do you see?

Well, there is a lion on that clock, but a cock on this clock.

Come on! It does not help.

What difference is there in the numerals?

The numbers are Arabic in that clock, but Roman in this clock.

This observation is helpful.

It is right that 78 is not divisible by 4. There is a trick in this problem.

The total sum of the sums on the four pieces is 80. The difference between 78 and 80 is 2. We need 2 extra to solve the problem.

In Roman Numerals IX stands for 9. If we separate it into X and I, we have 10 and 1. We have 2 extra.

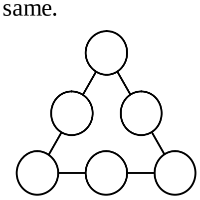
We may do the same thing to IV.

Now try to find a solution by yourself.

Creative Thinking Problems 25 to 27

25. Magic Triangle

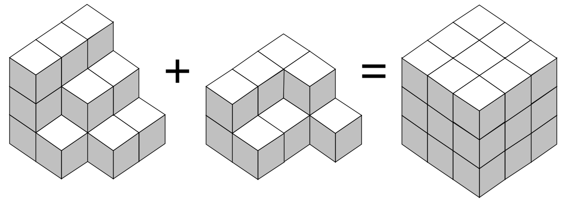
Fill 1 through 6 into 6 circles with a number in each circle such that the sum of three numbers in each side of the triangle is the same.



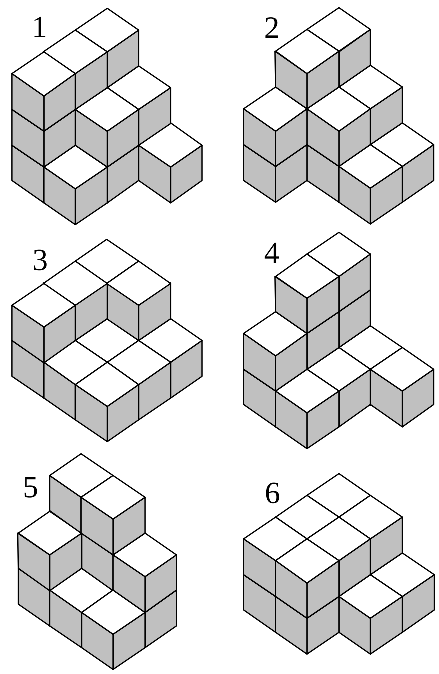
Consider two fillings to be the same if one can be overlapped with the other by rotating and/or flipping. Then there are four different fillings. Find all of them.

26. Making a Perfect Cube

The left two solids in the following figure can be assembled to make a perfect cube.

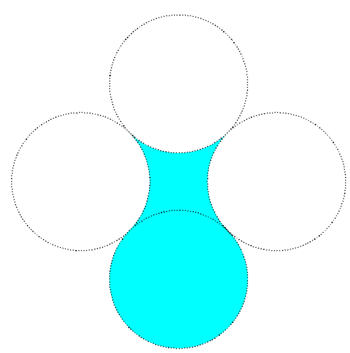


Which two solids below can be assembled to make a perfect cube?



27. Cutting the Vase Again

Look at the vase again.



This time cut the vase into three pieces and then assemble them to make a square.

(Clues and solutions will be given in the next issues.)

