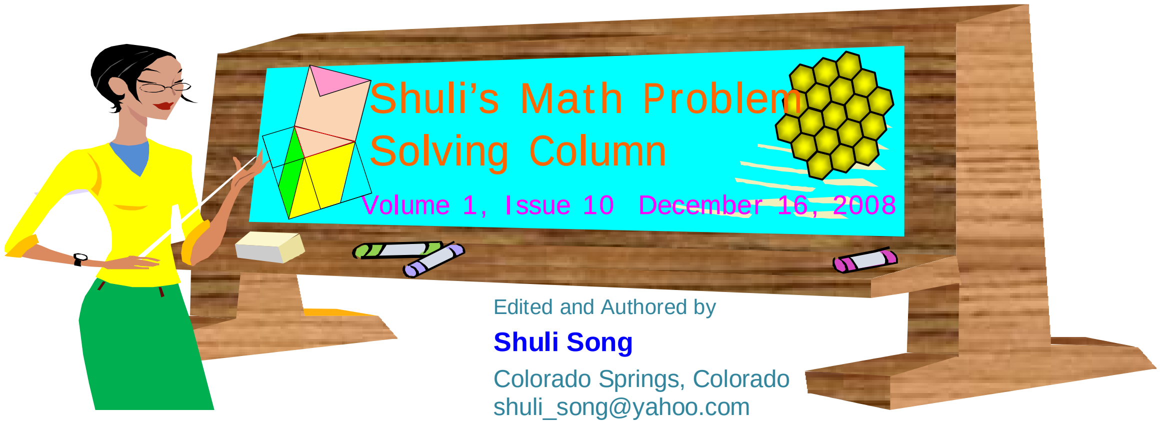


Edited and Authored by
Shuli Song
Colorado Springs, Colorado
shuli_song@yahoo.com

Content

Math Trick

Mental Calculation: $\overline{10a} \times \overline{10b}$

The Trick

Mentally calculate:

$107 \times 102 =$	$106 \times 105 =$	$103 \times 108 =$
$106^2 =$	$109 \times 104 =$	$104 \times 107 =$

Write these multiplications in the general form: $\overline{10a} \times \overline{10b}$ where *a* and *b* are digits.

A short cut to calculate the multiplications is shown through the following examples.

Example 1

Calculate 107×108.

Step 1: Calculate $\overline{10a} + b$.

In this example, $107 + 8 = 115$.

Step 2: Calculate $a \times b$.

In this example, $7 \times 8 = 56$.

Step 3: Attach $a \times b$ to the right of the result in step 1. In this example, attach 56 to the right of 115 : 11556.

Now we are done: $107 \times 108 = 11556$.

Example 2

Calculate 102×103.

Step 1: Calculate $102 + 3 = 105$.

Step 2: Calculate $2 \times 3 = 6$, treated as two digits: 06.

Step 3: Attach 06 to the right of 105: 10506.

We have $102 \times 103 = 10506$.

Why Does This Work?

Write $\overline{10a}$ and $\overline{10b}$ in the base 10 representation:

$$\overline{10a} = 100 + a \text{ and } \overline{10b} = 100 + b .$$

So we have

$$\begin{aligned}\overline{10a} \times \overline{10b} &= (100 + a) \times (100 + b) \\ &= 10000 + 100a + 100b + ab \\ &= 100(100 + a + b) + ab = 100(\overline{10a} + b) + ab.\end{aligned}$$

This shows that to calculate $\overline{10a} \times \overline{10b}$, we may do

Step 1: Calculate $\overline{10a} + b$.

Step 2: Calculate $a \times b$.

Step 3: Attach $a \times b$ as two digits to the right of $\overline{10a} + b$.

Practice Problems

$102 \times 109 =$	$103 \times 107 =$	$108 \times 104 =$
$107 \times 106 =$	$102 \times 105 =$	$106 \times 109 =$
$101 \times 109 =$	$108 \times 103 =$	$109 \times 108 =$
$103 \times 104 =$	$106 \times 105 =$	$107 \times 102 =$
$108^2 =$	$104^2 =$	$106^2 =$
$105^2 =$	$107^2 =$	$109^2 =$

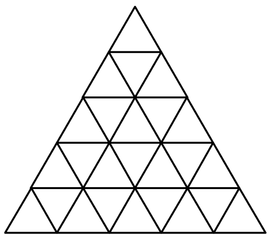
Math Competition Skill

How Do You Count? – Parallelograms in Triangular Grids

Problems

Problem 1

In the figure below, an equilateral triangle of side length 5 is divided into 25 small congruent equilateral triangles of side length 1. How many parallelograms of all sizes are there in the 5-layer diagram?



The equilateral triangles of side length 1 will be called *unit triangles.*

This is a fantastic counting problem.

It is very good for us to practice systematically listing according to numbers and shapes in solving counting problems.

In general, the problem can be presented as follows.

Problem 2

An equilateral triangle of side length *n* is divided into n^2 unit triangles. Find the number of parallelograms of all sizes in the *n*-layer diagram. The answer should be a formula in terms of *n*.

We will present a nice mathematical model in which we establish a one-to-one correspondence to obtain the general solution.

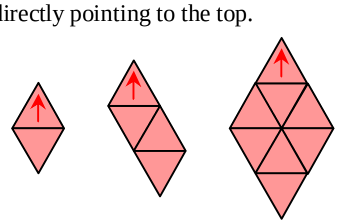
Solution to Problem 1

Let $f(n)$ be the number of parallelograms of all sizes if the original equilateral triangle has side length *n*.

We observe three kinds of parallelograms according to the orientations of parallelograms: 1. pointing to the top; 2. pointing to the left-bottom corner; 3. pointing to the right-bottom corner.

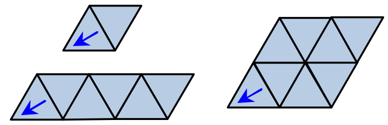
1. *Pointing to the top*

   In any of the three parallelograms below there is an angle directly pointing to the top.



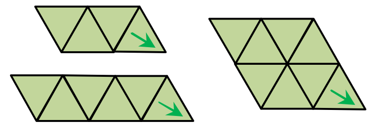
2. *Pointing to the left-bottom corner*

   Three parallelograms are illustrated for this type:



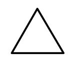
3. *Pointing to the right-bottom corner*

   Three parallelograms are shown for this type:



Now we start counting for $n = 1, 2, 3, 4, 5$.

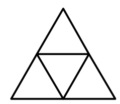
$n = 1$:



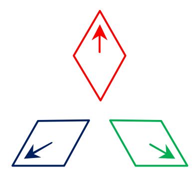
No parallelogram can be seen.

So $f(1) = 0$.

$n = 2$:

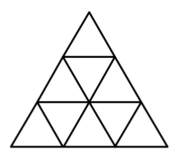


In each of the three directions there is one parallelogram:



So $f(2) = 3$.

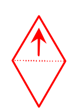
$n = 3$:



Because of symmetry, we will count only for the direction pointing to the top.

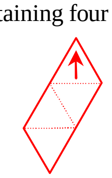
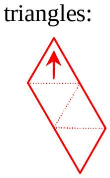
There are two different shapes of parallelograms categorized according to the number of unit triangles contained in the parallelograms.

Shape 1: Containing two unit triangles:



We readily see 3 parallelograms for this shape.

Shape 2: Containing four unit triangles:

 or 

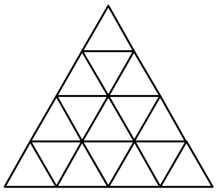
There are $2\times1=2$ parallelograms for this shape.

Altogether, there are $3+2=5$ parallelograms in this direction.

Therefore, there are $3\times5=15$ parallelograms in all three directions.

So $f(3)=15$.

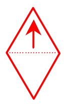
$n=4$:



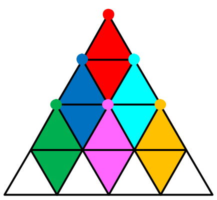
Again, we will count the parallelograms in the direction pointing to the top.

There are four different shapes of parallelograms.

Shape 1: Containing two unit triangles:

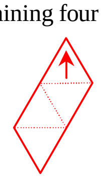


To count for this shape, I use the method called "*Using a Reference*" presented in *Issue 3, Volume 1*. We use the top vertex as the reference.

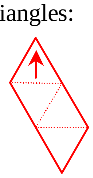


In the figure above all heavily marked points can be the reference. So there are 6 parallelograms for this shape, which are shaded with different colors.

Shape 2: Containing four unit triangles:

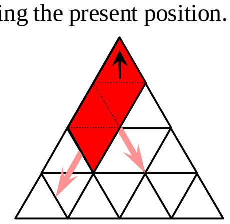


or



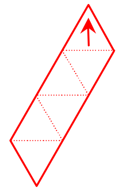
Because of symmetry, we count for the left shape. Now I use the method called "*Moving the Shape*" presented in *Issue 4, Volume 1*.

I put the shape at the top. We can move it in three positions including the present position.

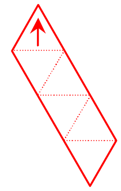


So there are 3 parallelograms for the left shape. Altogether, there are $2\times3=6$ parallelograms for this shape.

Shape 3: Containing six unit triangles:

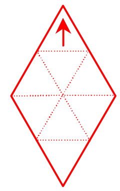


or



We see $2\times1=2$ parallelograms for this shape.

Shape 4: Containing eight unit triangles:



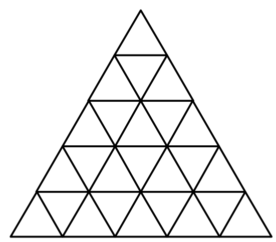
There is only one parallelogram for this shape.

Altogether, there are $6+6+2+1=15$ parallelograms in this direction.

Therefore, there are $3\times15=45$ parallelograms in all three directions.

Thus $f(4)=45$.

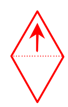
$n=5$:



We will also count for the direction pointing to the top.

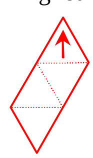
There are six different shapes of parallelograms:

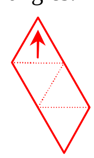
Shape 1: Containing two unit triangles:



We observe 10 parallelograms for this shape.

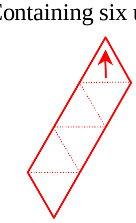
Shape 2: Containing four unit triangles:

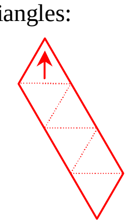




For this shape there are $2\times6=12$ parallelograms.

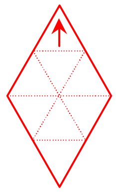
Shape 3: Containing six unit triangles:





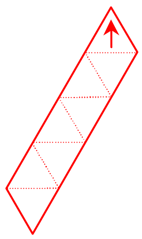
We can count $2\times3=6$ parallelograms.

Shape 4: Containing eight unit triangles:

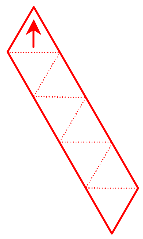


We see 3 parallelograms for this shape.

Shape 5: Containing eight unit equilateral triangles of the second type:

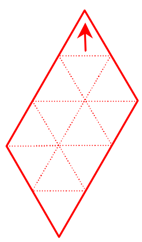


or

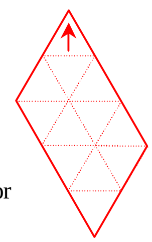


There are $2\times 1=2$ parallelograms for this shape.

Shape 6: Containing twelve unit triangles:



or



We observe $2\times 1=2$ parallelograms for this shape.

Altogether, there are

$$10+12+6+3+2+2=35$$

parallelograms in this direction.

Therefore, there are $3\times 35=105$ parallelograms in all three directions.

Hence $f(5)=105$.

Solution to Problem 2

It is not easy to find a formula as the solution to the general problem.

In fact, the number of parallelograms in the *n*-layer diagram is

$$f(n)=3\cdot\binom{n+2}{4}.$$

If we check it,

$$f(1)=3\cdot\binom{3}{4}=0;$$

$$f(2)=3\cdot\binom{4}{4}=3;$$

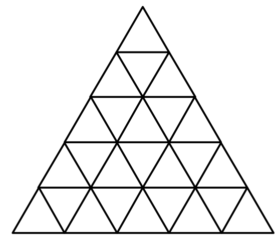
$$f(3)=3\cdot\binom{5}{4}=3\cdot\binom{5}{1}=3\cdot 5=15;$$

$$f(4)=3\cdot\binom{6}{4}=3\cdot\binom{6}{2}=3\cdot\frac{6\times 5}{2\times 1}=45;$$

$$f(5)=3\cdot\binom{7}{4}=3\cdot\binom{7}{3}=3\cdot\frac{7\times 6\times 5}{3\times 2\times 1}=105,$$

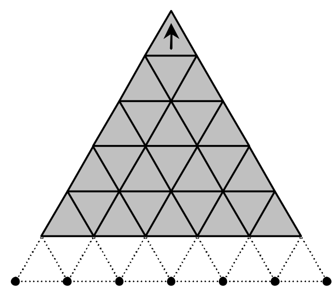
it is correct for $n=1, 2, 3, 4, 5$.

We will derive it by using the following 5-layer diagram.



Again, we will count the parallelograms in the direction pointing to the top.

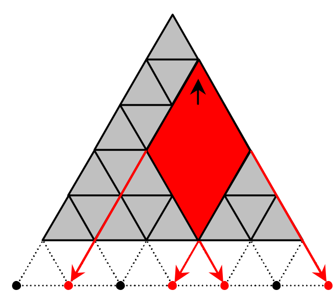
Attach one layer of unit triangles at the bottom as shown.



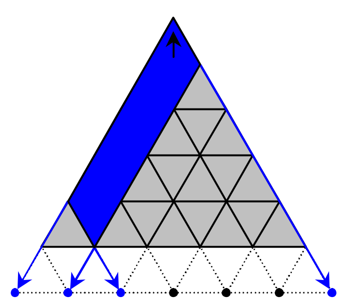
Seven points are heavily marked. Note that $7=5+2$.

We claim that there is a one-to-one correspondence between the parallelograms and the selections of four points from the seven points.

For example, the red parallelogram corresponds to the four red points:

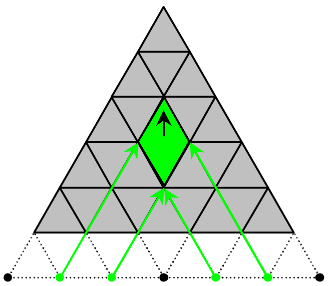


The blue parallelogram corresponds to the four blue points:

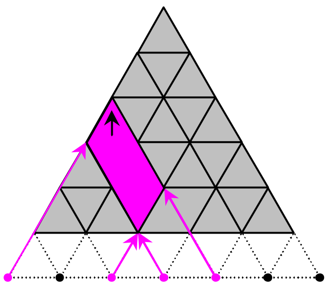


The reverse is also true.

The four green points uniquely determine the green parallelogram:



The pink four points uniquely determine the pink parallelogram:



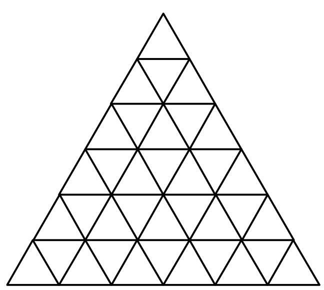
Therefore, the number of parallelograms in one direction in the 5-layer diagram is $\binom{7}{4}$. Then the number of parallelograms in all three directions is $3 \cdot \binom{7}{4}$.

In general, the number of parallelograms in the *n*-layer diagram is

$$f(n) = 3 \cdot \binom{n+2}{4}.$$

Practice Problem

Practice systematically listing to find $f(6)$.



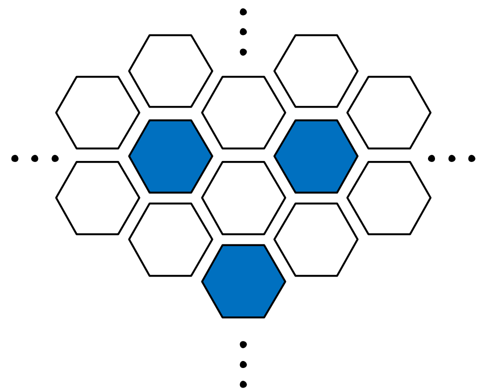
A Problem from a Real Math Competition

Today's problem comes from Canadian Mathematics Competition (CMC).

Problem

(CMC 2000 Grade 7 Gauss Problem 25, Grade 8 Gauss Problem 23)

An infinitely large floor is tiled, as partially shown, with regular hexagonal tiles. The tiles are colored blue or white. Each blue tile is surrounded by 6 white tiles and each white tile is surrounded by 3 white and 3 blue tiles. Find the ratio of the number of blue tiles to the number of white tiles.

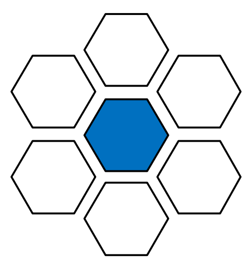


Note that the original problem has been revised slightly.

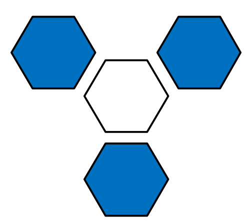
Answer: $1:2$

Solution:

Look at a blue tile. There are 6 white tiles around it. If we just look at these 7 tiles shown below, the ratio of the number of blue tiles to the number of white tiles appears to be $1:6$.



However, each white tile is adjacent to three 3 blue tiles, as shown:



That is, each white tile is counted three times by the three blue tiles around it respectively.

Note that $6 \div 3 = 2$. So the ratio of the number of blue tiles to the number of white tiles is $1:2$.

Practice Problem

A soccer ball is constructed using regular pentagons and hexagons with equal side lengths for its surface, as shown below. If there are 12 pentagons, how many hexagons are there?

Answers to All Practice Problems in Last Issue

Math Trick: Mental Calculation

Practice Problems I

1911	2201	3321
4331	1071	5551
1001	2511	7371
1271	3111	1491
6561	1681	961
2601	5041	8281

Practice Problems II

7471	18491	90181
45811	56871	55571
37851	110691	15851
563101	339281	556491
32661	251121	203401

Practice Problems III

1369851	101331	1002631
6507601	11909401	80118801

Chickens and Rabbits

Practice Problems I

1. 8 2. 31.5

Practice Problems II

1. 6 2. 21 3. 18
4. E 5. E

A Problem from a Real Math Competition

68

Solutions to Creative Thinking Problems 22 to 24

22. Route of 100

Consider the route with the maximum sum:

1	1	1	1	1	1	1	1	1
2	2	2	2	2	2	2	2	2
3	3	3	3	3	3	3	3	3
4	4	4	4	4	4	4	4	4
5	5	5	5	5	5	5	5	5
6	6	6	6	6	6	6	6	6
7	7	7	7	7	7	7	7	7
8	8	8	8	8	8	8	8	8
9	9	9	9	9	9	9	9	9

The sum of the numbers in the route is

$$1+2+\cdots+9+8\times 9=117.$$

If we change the route to the route shown below, the sum is decreased by 8.

1	1	1	1	1	1	1	1	1
2	2	2	2	2	2	2	2	2
3	3	3	3	3	3	3	3	3
4	4	4	4	4	4	4	4	4
5	5	5	5	5	5	5	5	5
6	6	6	6	6	6	6	6	6
7	7	7	7	7	7	7	7	7
8	8	8	8	8	8	8	8	8
9	9	9	9	9	9	9	9	9

If we change the route further as shown, the sum is decreased by another 8.

1	1	1	1	1	1	1	1	1
2	2	2	2	2	2	2	2	2
3	3	3	3	3	3	3	3	3
4	4	4	4	4	4	4	4	4
5	5	5	5	5	5	5	5	5
6	6	6	6	6	6	6	6	6
7	7	7	7	7	7	7	7	7
8	8	8	8	8	8	8	8	8
9	9	9	9	9	9	9	9	9

The sum in this route is $117-2\times 8=101$. We still have to deduct 1 from it. By changing the route as follows, we obtain a solution.

1	1	1	1	1	1	1	1	1
2	2	2	2	2	2	2	2	2
3	3	3	3	3	3	3	3	3
4	4	4	4	4	4	4	4	4
5	5	5	5	5	5	5	5	5
6	6	6	6	6	6	6	6	6
7	7	7	7	7	7	7	7	7
8	8	8	8	8	8	8	8	8
9	9	9	9	9	9	9	9	9

Starting from the route with the minimum sum:

1	1	1	1	1	1	1	1	1
2	2	2	2	2	2	2	2	2
3	3	3	3	3	3	3	3	3
4	4	4	4	4	4	4	4	4
5	5	5	5	5	5	5	5	5
6	6	6	6	6	6	6	6	6
7	7	7	7	7	7	7	7	7
8	8	8	8	8	8	8	8	8
9	9	9	9	9	9	9	9	9

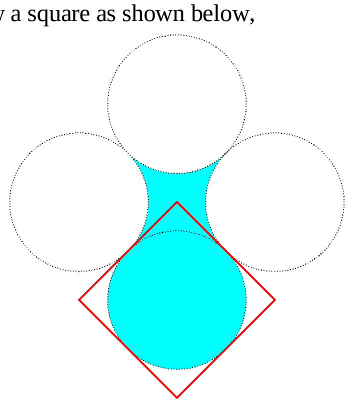
we can obtain a solution similarly.

In fact, if we know "gradually tuning up", we can start with any route such as:

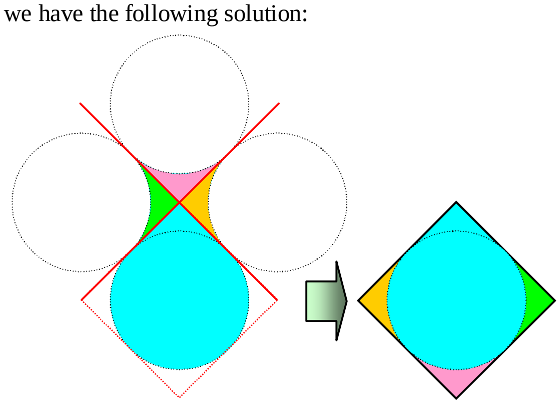
1	1	1	1	1	1	1	1	1
2	2	2	2	2	2	2	2	2
3	3	3	3	3	3	3	3	3
4	4	4	4	4	4	4	4	4
5	5	5	5	5	5	5	5	5
6	6	6	6	6	6	6	6	6
7	7	7	7	7	7	7	7	7
8	8	8	8	8	8	8	8	8
9	9	9	9	9	9	9	9	9

23. Cutting the Vase

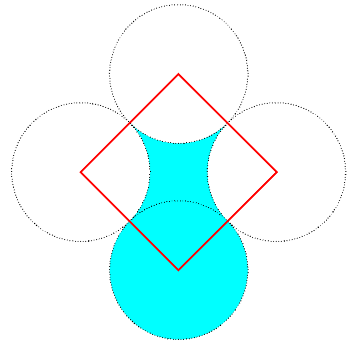
If we draw a square as shown below,



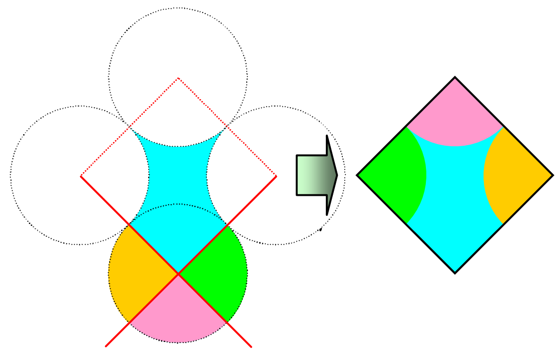
we have the following solution:



If we draw a square this way:



we have the second solution:

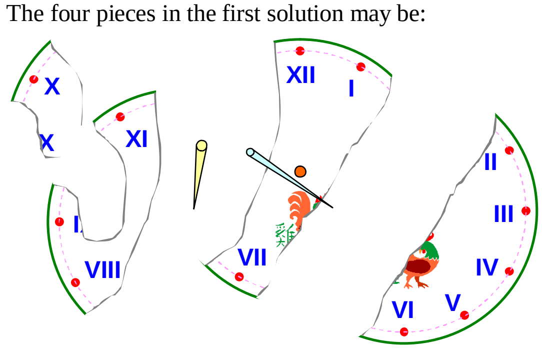


24. Broken Clock Again

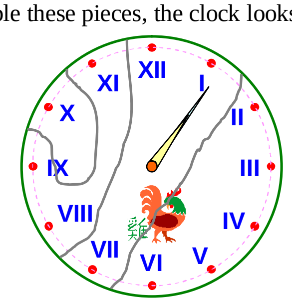
Since $1+2+\cdots+12=78$ is not equal to $20\times 4=80$, we have to generate 2 extra.

We have two ways to do this: 1. separating IX (9) into X (10) and I (1); 2. separating IV (4) into V (5) and I (1). So we have two solutions.

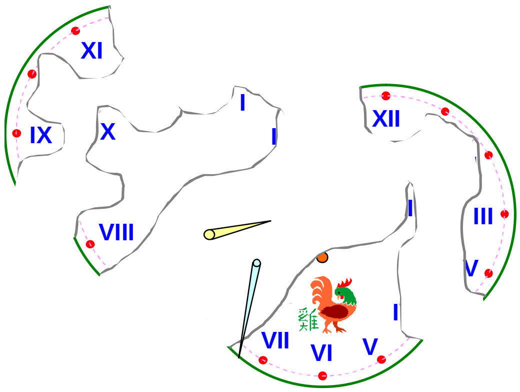
The four pieces in the first solution may be:



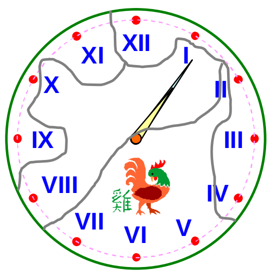
If we assemble these pieces, the clock looks like:



The following four pieces may be the second solution:



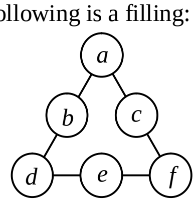
After assembly the clock appears as:



Clues to Creative Thinking Problems 25 to 27

25. Magic Triangle

Assume that the following is a filling:



where a, b, c, d, e, and f are 1 to 6 in some order. Let M be the magic sum. That is,

$$a+b+d=M\ ,\ d+e+f=M\ ,\ f+c+a=M\ .$$

Adding them we have

$$a+b+c+d+e+f+a+d+f=3M\ .$$

Note that $a+b+c+d+e+f=21$. We obtain

$$a+d+f=3M-21=3(M-7).$$

It follows that the sum of the three numbers filled at the vertices must be divisible by 3.

26. Making a Cube

Count the unit cubes in each solid first.

27. Cutting the Vase Again

You had experiences to cut the vase into four pieces. Again, draw an appropriate square to find the solution.

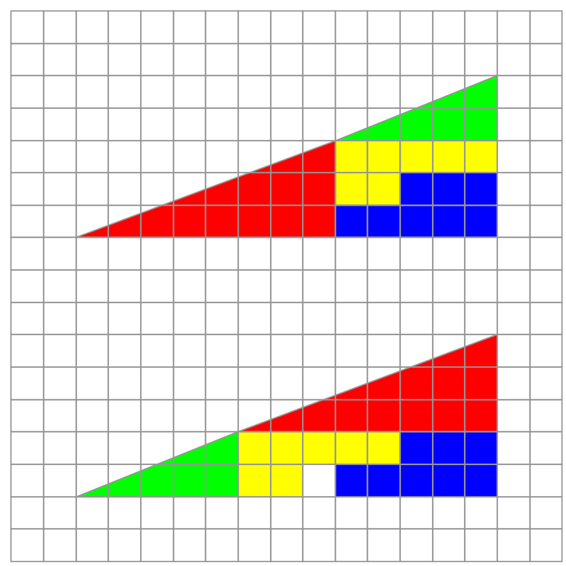
Creative Thinking Problems 28 to 30

28. Cutting a Birthday Cake

Can you cut a birthday cake into 8 pieces with three straight cuts?

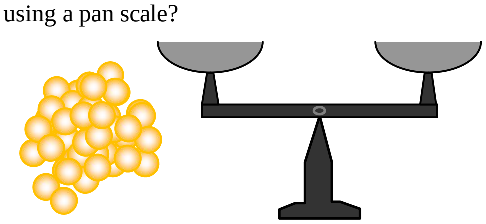
29. Where does the Hole Come?

I cut the shape at the top in the figure into four pieces and then assemble them into a new shape at the bottom. The new shape looks the same as the original one, but there is a hole in it. Where does the hole come?



30. 32 Balls

There are 32 balls that look exactly the same. Among them 30 are golden and 2 are fake. All golden balls have the same weight, while the two fake balls have the same weight, which is slightly different from that of the golden balls. Now two brothers are going to divide the gold equally. At least how many times of weighing do they need to divide the balls into two groups of equal total weight (to guarentee the same amount of gold for each) by using a pan scale?



(Clues and solutions will be given in the next issues.)

