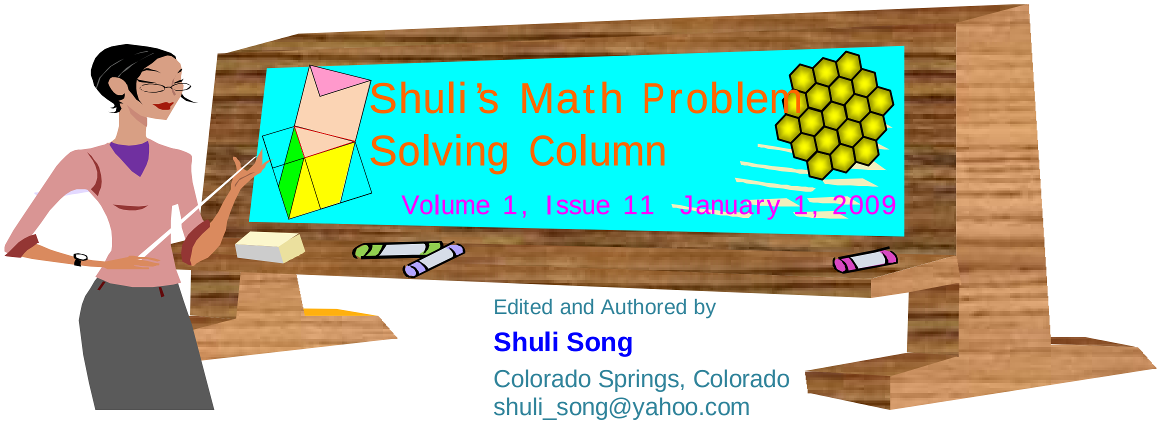


Content

Math Trick

Mental Calculation: $\overline{2a}\times\overline{2b}$

The Trick

Mentally calculate:

$27\times 22=$	$26\times 25=$	$23\times 28=$
$26^2=$	$29\times 24=$	$24\times 27=$

Write these multiplications in the general form: $\overline{2a}\times\overline{2b}$ where a and b are digits.

The short cut to do the multiplications is shown through the following examples.

Example 1

Calculate 26×23.

Step 1: Calculate $\overline{2a}+b$.

In this example, $26+3=29$.

Step 2: Multiply the result in step 1 by 2.

In this example, $29\times 2=58$.

Step 3: Calculate $a\times b$.

In this example, $6\times 3=18$.

Step 4 "Add" them this way:

$$\begin{array}{cccc} & 5 & 8 & \\ + & & 1 & 8 \\ \hline & 5 & 9 & 8 \end{array}$$

We are done: $26\times 23=598$.

Example 2

Calculate 25×29.

Step 1: Calculate $25+9=34$.

Step 2: Calculate $34\times 2=68$.

Step 3: Calculate $5\times 9=45$.

Step 4: "Add" them:

$$\begin{array}{cccc} & 6 & 8 & \\ + & & 4 & 5 \\ \hline & 7 & 2 & 5 \end{array}$$

We have $25\times 29=725$.

Why Does This Work?

Write $\overline{2a}$ and $\overline{2b}$ in the base 10 representation:

$$\overline{2a}=20+a \text{ and } \overline{2b}=20+b.$$

So we have

$$\overline{2a}\times\overline{2b}=(20+a)\times(20+b)=400+20a+20b+ab$$
$$=20\times(20+a+b)+ab=20\times(\overline{2a}+b)+ab.$$

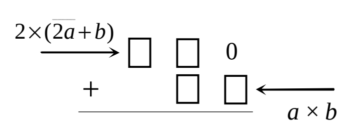
This shows that to calculate $\overline{2a}\times\overline{2b}$, we may do

Step 1: Calculate $\overline{2a}+b$.

Step 2: Multiply the result in step 1 by 2.

Step 3: Calculate $a\times b$.

Step 4: "Add" them this way:



Mental Calculation: $\overline{na} \times \overline{nb}$

The similar procedure applies to the multiplications in the form $\overline{na} \times \overline{nb}$ where n is a digit greater than 2. Instead of 2 we have to multiply $\overline{na} + b$ by n in step 2.

Example 3

Calculate 34×38.

Step 1: Calculate $34 + 8 = 42$.

Step 2: Calculate $42 \times 3 = 126$.

Step 3: Calculate $4 \times 8 = 32$.

Step 4: "Add" them:

	1	2	6	
+			3	2
	1	2	9	2

We obtain $34 \times 38 = 1292$.

Example 4

Calculate 45×47.

Step 1: Calculate $45 + 7 = 52$.

Step 2: Calculate $52 \times 4 = 208$.

Step 3: Calculate $5 \times 7 = 35$.

Step 4: "Add" them:

	2	0	8	
+			3	5
	2	1	1	5

Then $45 \times 47 = 2115$.

Example 5

Calculate 76×72.

Step 1: Calculate $76 + 2 = 78$.

Step 2: Calculate $78 \times 7 = 546$.

Step 3: Calculate $6 \times 2 = 12$.

Step 4: "Add" them:

	5	4	6	
+			1	2
	5	4	7	2

Then $76 \times 72 = 5472$.

Practice Problems I

$24 \times 29 =$	$23 \times 27 =$	$28 \times 24 =$
$27 \times 26 =$	$22 \times 25 =$	$26 \times 29 =$
$23 \times 24 =$	$28 \times 23 =$	$27 \times 21 =$
$21^2 =$	$24^2 =$	$23^2 =$
$25^2 =$	$27^2 =$	$29^2 =$
$28^2 =$	$22^2 =$	$26^2 =$

Practice Problems II

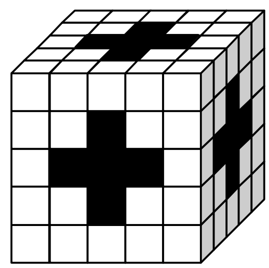
$32 \times 39 =$	$33 \times 37 =$	$38 \times 34 =$
$47 \times 46 =$	$42 \times 45 =$	$46 \times 49 =$
$51 \times 59 =$	$58 \times 53 =$	$69 \times 68 =$
$73 \times 74 =$	$86 \times 85 =$	$97 \times 92 =$

Math Competition Skill

How Do You Count? – Surface Area of Cubes

Problem

A $5 \times 5 \times 5$ cube is formed using $1 \times 1 \times 1$ cubes. A number of the smaller cubes are removed by punching out the 15 designated columns from front to back, top to bottom, and side to side.



We are familiar with this solid. We have counted the number of unit cubes remaining in section "A Problem from a Real Math Competition" of *Issue 9, Volume 1*.

In this short lesson we will count the total surface area (both external and internal).

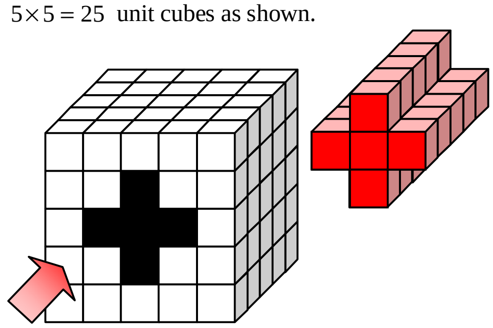
Solution

I will present four solutions. It is not easy to see solutions one and two if we don't draw. It may not be so difficult to find solution three. However, solution four is elegant. You should use this method to solve the practice problems.

Solution One:

We start with the complete $5 \times 5 \times 5$ cube without any punches. The total surface area of it is $6 \times 5^2 = 150$ square units.

First we punch from front to back. We remove $5 \times 5 = 25$ unit cubes as shown.

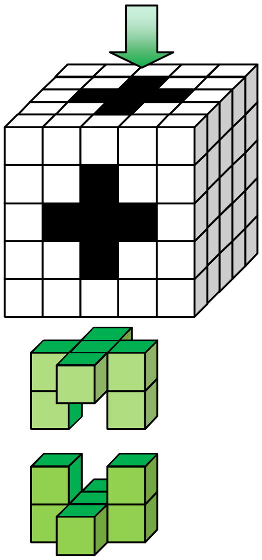


With this punch the surfaces colored with red disappear from the starting $5 \times 5 \times 5$ cube, and the surfaces colored with light red are generated as the internal surfaces in the new solid. The net increase in the total surface area is

$$12 \times 5 - 2 \times 5 = 50.$$

Then we punch from top to bottom.

This time we remove another $5+2+0+2+5=14$ unit cubes:

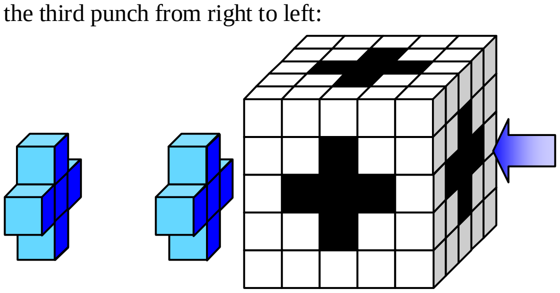


With this punch the surfaces colored with green disappear from the solid after the first punch, and the surfaces colored with light green are generated. The net increase in the total surface area is

$$2\times(2\times5+2\times4-2\times5-2\times1)=12 .$$

At last we punch from right to left.

We remove $5+0+0+0+5=10$ unit cubes more with the third punch from right to left:



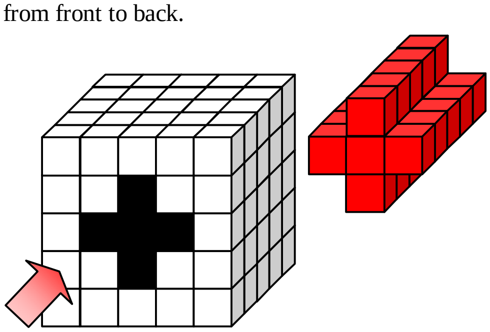
With this punch the surfaces colored with blue disappear from the solid after the first two punches, and the surfaces colored with light blue are generated. The net increase in the total surface area is

$$2\times(4\times3-2\times5)=4 .$$

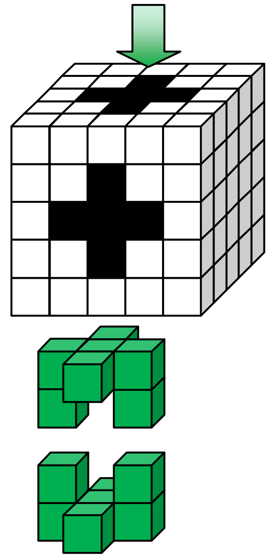
So the total surface area is $150+50+12+4=216$.

Solution Two:

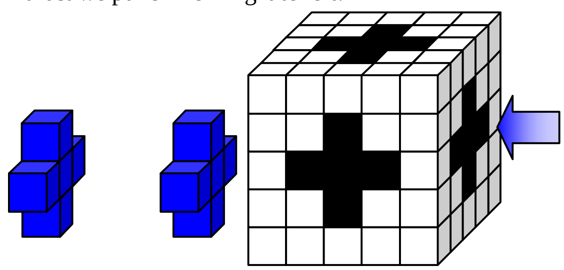
Start with the complete $5\times5\times5$ cube. First we punch from front to back.



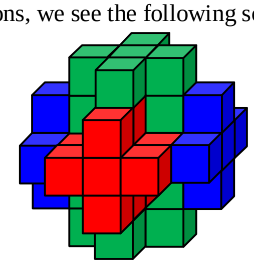
Then we punch from top to bottom:



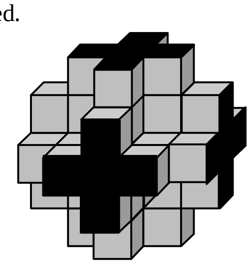
At last we punch from right to left:



After we assemble all the removed unit cubes with their original positions, we see the following solid.



As shown below, the black surfaces are parts of surfaces of the starting $5\times5\times5$ cube, and the gray surfaces are newly generated.

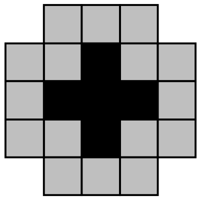


The gray surfaces of the above solid actually touch the internal surfaces of the original punched solid (given solid in the problem) before they are separated from the starting complete $5\times5\times5$ cube.

Now we count the total surface areas for both black and gray parts of the above solid.

How do you count?

If we project this solid in any direction: front, back, top, bottom, right, or left, we will see the following figure.



The area of the black squares is 5, and the area of the gray squares is 16. Then the total area of the black surfaces is $6 \times 5 = 30$, and the total area of the gray surfaces is $16 \times 6 = 96$.

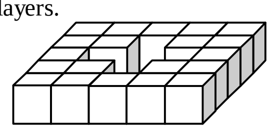
The total surface area in the starting omplete $5 \times 5 \times 5$ cube is $6 \times 5^2 = 150$.

Therefore, the total surface area in the original punched solid is

$$150 - 30 + 96 = 216.$$

Solution Three:

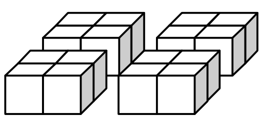
There are five layers of unit cubes counting from top to bottom. The figure below shows the unit cubes in the first and fifth layers.



The total surface area in one of these two layers is

$$20 + 4 \times 5 + 4 \times 3 + 4 \times 1 = 56.$$

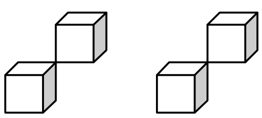
The figure below shows the unit cubes in the second and fourth layers.



The total surface area in one of these two layers is

$$4 \times (4 \times 2 + 3) = 44.$$

The figure below shows the unit cubes in the third layer.



The total surface area in this layer is

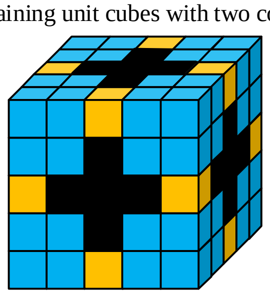
$$4 \times 4 = 16.$$

So the total surface area is

$$56 + 44 + 16 + 44 + 56 = 216.$$

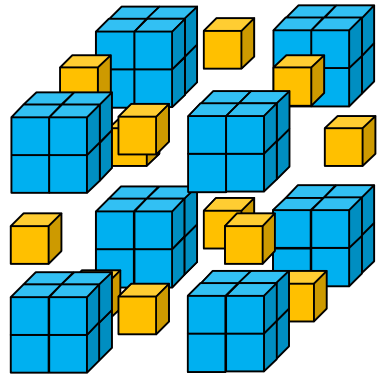
Solution Four:

I color all remaining unit cubes with two colors:



What did you see?

I separate the orange cubes from the blue cubes.



We see eight $2 \times 2 \times 2$ cubes (blue) at the corners of the original solid and twelve unit cubes (orange) at the edges of it.

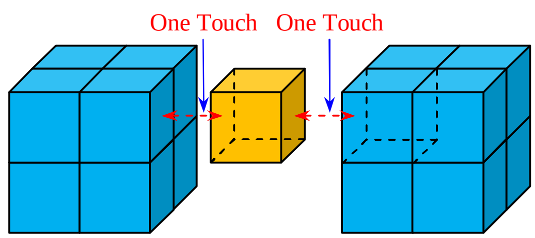
The surface area of a $2 \times 2 \times 2$ cube is $6 \times 2^2 = 24$.

The surface area of a $1 \times 1 \times 1$ cube is $6 \times 1^2 = 6$.

After the separation, the total surface area in the eight $2 \times 2 \times 2$ cubes and twelve $1 \times 1 \times 1$ cubes is

$$8 \times 24 + 12 \times 6 = 264.$$

Now we glue the eight $2 \times 2 \times 2$ cubes and twelve $1 \times 1 \times 1$ cubes together to form the original solid.



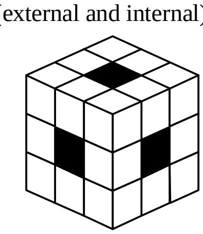
Every $1 \times 1 \times 1$ cube has two touches to two $2 \times 2 \times 2$ cubes. The number of total touches is $2 \times 12 = 24$. One touch makes the total surface area decrease by 2.

Therefore, the total surface area is:

$$264 - 24 \times 2 = 216.$$

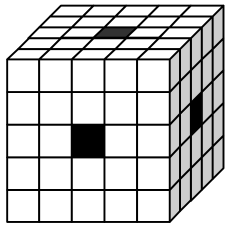
Practice Problems

1. A $3 \times 3 \times 3$ cube is formed using $1 \times 1 \times 1$ cubes. A number of the smaller cubes are removed by punching out the 3 designated columns from front to back, top to bottom, and right to left. Find the total surface area (external and internal) in square units.

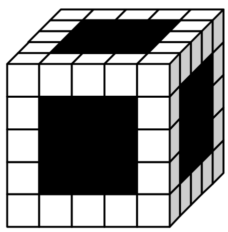


2. A $5 \times 5 \times 5$ cube is formed using $1 \times 1 \times 1$ cubes. A number of the smaller cubes are removed by punching out the 3 designated columns from front to

back, top to bottom, and right to left. Find the total surface area (external and internal) in square units.



3. A $5\times5\times5$ cube is formed using $1\times1\times1$ cubes. A number of the smaller cubes are removed by punching out the 27 designated columns from front to back, top to bottom, and right to left. Find the total surface area (external and internal) in square units.



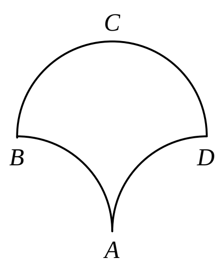
A Problem from a Real Math Competition

Today's problem comes from American Mathematics Contest Grade 8 (AMC8).

Problem

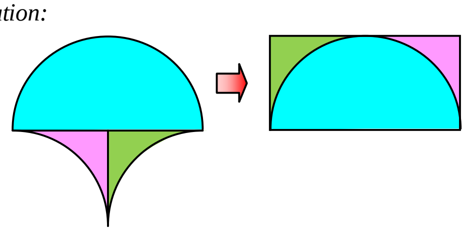
(16th AMC8 2000 Problem 19)

Three circular arcs of radius 5 units bound the region shown. Arcs *AB* and *AD* are quarter-circles, and arc *BCD* is a semicircle. What is the area, in square units, of the region?



Answer: $5\times10=50$

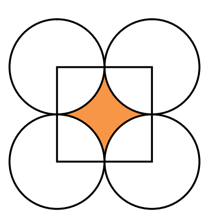
Solution:



Practice Problem

(8th AMC8 1992 Problem 24)

Four circles of radius 3 are arranged as shown. Their centers are the vertices of a square. Find the area of the shaded region.



Answers to All Practice Problems in Last Issue

Math Trick: Mental Calculation

11118	11021	11232
11342	10710	11554
11009	11124	11772
10712	11130	10914
11664	10816	11236
11025	11449	11881

How Do You Count? - Parallelograms in Triangular Grids

$f(6)=210$

A Problem from a Real Math Competition

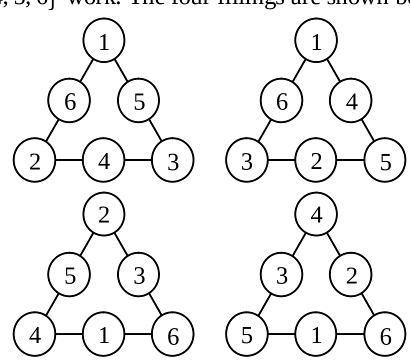
20

Solutions to Creative Thinking Problems 25 to 27

25. Magic Triangle

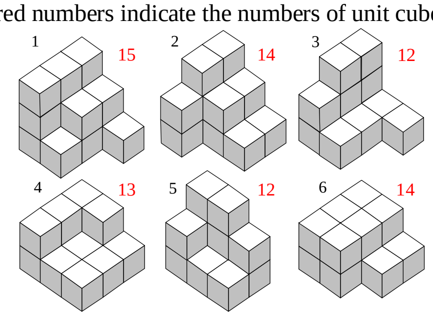
Divide the set $\{1, 2, 3, 4, 5, 6\}$ into three subsets: $\{1, 4\}$, $\{2, 5\}$, $\{3, 6\}$. Select three numbers with one from each subset. Then the sum of the three numbers is divisible by 3. We should fill these three numbers at the vertices.

There are 8 ways to choose three number: $\{1, 2, 3\}$, $\{1, 2, 6\}$, $\{1, 3, 5\}$, $\{1, 5, 6\}$, $\{2, 3, 4\}$, $\{2, 4, 6\}$, $\{3, 4, 5\}$, $\{4, 5, 6\}$. Only $\{1, 2, 3\}$, $\{1, 3, 5\}$, $\{2, 4, 6\}$, and $\{4, 5, 6\}$ work. The four fillings are shown below:

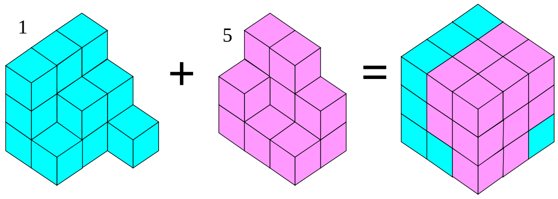


26. Making a Perfect Cube

The red numbers indicate the numbers of unit cubes.

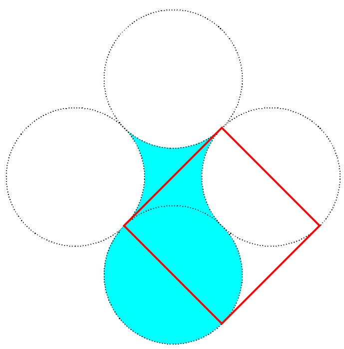


Focus on $15+12=27$ and $14+13=27$. We easily see that solid 1 and solid 5 can be assembled to make a perfect cube:

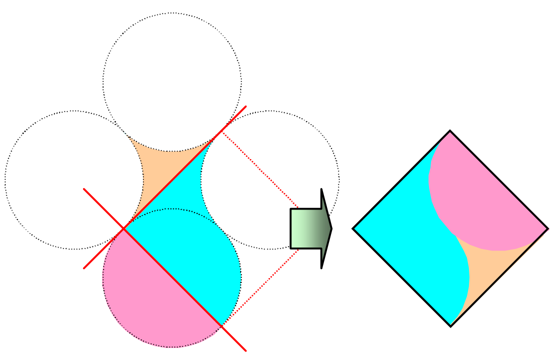


27. Cutting the Vase Again

If we draw a square this way:



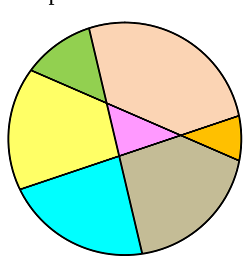
we have the solution:



Clues to Creative Thinking Problems 28 to 30

28. Cutting Birthday Cake

As shown in the figure below, we can generate at most 7 pieces with three straight cuts if you treat the cake as a two-dimensional shape.



29. Where does the Hole Come?

Pay attention to the red and green pieces, which are truly triangles.

30. 32 Balls

Note that $32=2^5$. Divide the balls into two equal groups of 16 balls each. Weigh one group against the other group.

If the scale is in balance, we are done.

If it is not, exchange 8 balls between the two groups.

Now you go ahead.

Creative Thinking Problems 31 to 33

31. True or False

Exactly one of the following five statements is true. Which one?

1. All of the following are true.
2. None of the following is true.
3. Some of the following are true.
4. All of the above are true.
5. None of the above is true.

32. We Miss 2008

Example

Place one operation symbol, selected from +, –, ×, and ÷, or no symbol before each number in the left side such that the expression becomes true:

1 2 3 4 5 = 24.

No parentheses are allowed.

I have two solutions for the above expression:

– 1 + 2 + 3 + 4 × 5 = 24;

1 2 + 3 + 4 + 5 = 24.

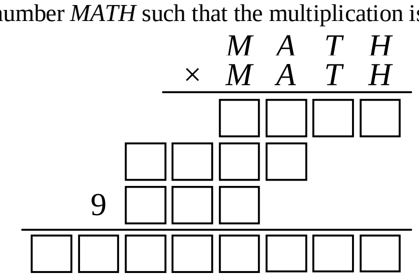
In the second solution, there is no symbol between 1 and 2. Then they are treated as 12.

Year 2008 has just passed. We miss 2008. Let us make the following expression true:

1 2 3 4 5 6 7 8 9 = 2008.

33. MATH

Different letters represent different digits. Find the four-digit number *MATH* such that the multiplication is true.



(Clues and solutions will be given in the next issues.)

