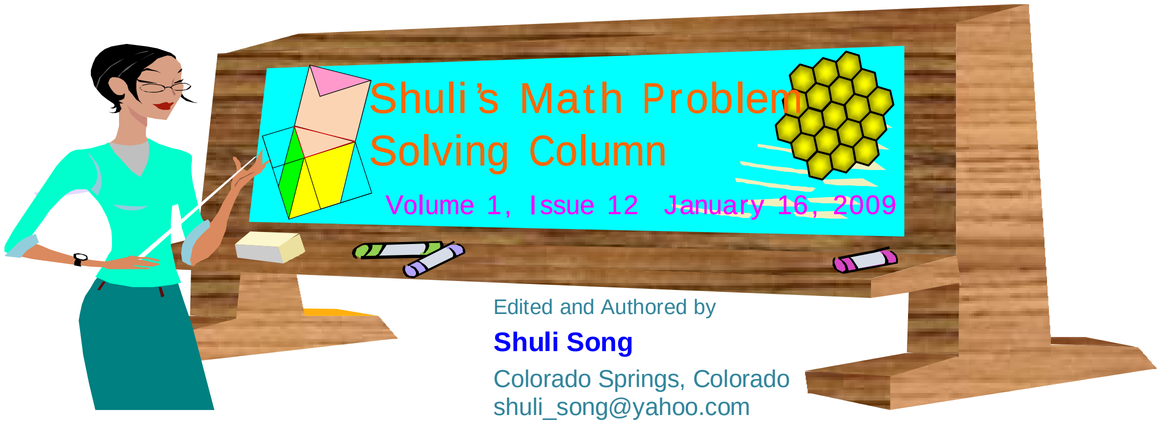

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Math Trick

Mental Calculation: $\overline{a2} \times \overline{b2}$

The Trick

Mentally calculate:

$72 \times 22 =$ $\quad$ $62 \times 52 =$ $\quad$ $32 \times 82 =$

$62^2 =$ $\quad$ $92 \times 42 =$ $\quad$ $42 \times 72 =$

Write these multiplications in the general form: $\overline{a2} \times \overline{b2}$ where a and b are digits. The short cut to calculate the multiplications is shown through the examples below.

Example 1

Calculate 62×72.

Step 1: Calculate $a \times b$. In this example, $6 \times 7 = 42$.

Step 2: Calculate $a + b$. In this example, $6 + 7 = 13$.

Step 3: Multiply the result in step 2 by 2. In this example, $2 \times 13 = 26$.

Step 4: "Add" $a \times b$ and $2 \times (a + b)$ this way:

$$\begin{array}{cccc} & 4 & 2 & \\ + & & 2 & 6 \\ \hline & 4 & 4 & 6 \end{array}$$

Step 5: Attach $2^2 = 4$ to the right of the result in step 4: 4464.

We are done: $62 \times 72 = 4464$.

Example 2

Calculate 52×32.

Step 1: Calculate $5 \times 3 = 15$.

Step 2: Calculate $5 + 3 = 8$.

Step 3: Calculate $2 \times 8 = 16$.

Step 4: "Add" 15 and 16:

$$\begin{array}{cccc} & 1 & 5 & \\ + & & 1 & 6 \\ \hline & 1 & 6 & 6 \end{array}$$

Step 5: Attach 4: 1664.

We have $52 \times 32 = 1664$.

It works for the numbers with three or more digits.

Example 3

Calculate 252×62.

Step 1: Calculate $25 \times 6 = 150$.

Step 2: Calculate $25 + 6 = 31$.

Step 3: Calculate $2 \times 31 = 62$.

Step 4: "Add" 150 and 62:

$$\begin{array}{ccccc} & 1 & 5 & 0 & \\ + & & & 6 & 2 \\ \hline & 1 & 5 & 6 & 2 \end{array}$$

Step 5: Attach 4: 15624.

We obtain $252 \times 62 = 15624$.

Why Does This Work?

Write $\overline{a2}$ and $\overline{b2}$ in the base 10 representation:

$$\overline{a2} = 10a + 2 \text{ and } \overline{b2} = 10b + 2.$$

So we have

$$\overline{a2} \times \overline{b2} = (10a + 2) \times (10b + 2) = 100ab + 20a + 20b + 4$$
$$= 10 \times [10ab + 2 \times (a + b)] + 4.$$

This shows that to calculate $\overline{a2} \times \overline{b2}$, we may do

Step 1: Calculate $a \times b$.

Step 2: Calculate $a + b$.

Step 3: Calculate $2 \times (a+b)$.

Step 4: "Add" $a \times b$ and $2 \times (a+b)$ this way: $10ab + 2 \times (a+b)$.

Step 5: Attach $2^2 = 4$ to the right of the result in step 4.

Mental Calculation: $\overline{an} \times \overline{bn}$

The similar procedure applies to the multiplications in the form $\overline{an} \times \overline{bn}$ where n is a digit greater than 2. Instead of 2 we have to multiply $a+b$ by n in step 3. Instead of $2^2 = 4$ we have to attach n^2 in step 5. When n^2 has two digits, we actually attach the ones digit of n^2, and carry the tens digit to the result in step 4.

Example 4

Calculate 43×83.

Step 1: Calculate $4 \times 8 = 32$.

Step 2: Calculate $4 + 8 = 12$.

Step 3: Calculate $3 \times 12 = 36$.

Step 4: "Add" 32 and 36:

$$\begin{array}{rccc} & 3 & 2 & \\ + & & 3 & 6 \\ \hline & 3 & 5 & 6 \end{array}$$

Step 5: Attach $3^2 = 9$: 3569.

We obtain $43 \times 83 = 3569$.

Example 5

Calculate 54×94.

Step 1: Calculate $5 \times 9 = 45$.

Step 2: Calculate $5 + 9 = 14$.

Step 3: Calculate $4 \times 14 = 56$.

Step 4: "Add" 45 and 56:

$$\begin{array}{rccc} & 4 & 5 & \\ + & & 5 & 6 \\ \hline & 5 & 0 & 6 \end{array}$$

Step 5: "Attach" $4^2 = 16$: 5076.

Then $54 \times 94 = 5076$.

Example 6

Calculate 67×27.

Step 1: Calculate $6 \times 2 = 12$.

Step 2: Calculate $6 + 2 = 8$.

Step 3: Calculate $7 \times 8 = 56$.

Step 4: "Add" 12 and 56:

$$\begin{array}{rccc} & 1 & 2 & \\ + & & 5 & 6 \\ \hline & 1 & 7 & 6 \end{array}$$

Step 5: "Attach" $7^2 = 49$: 1809

Then $67 \times 27 = 1809$.

Practice Problems I

$42 \times 92 =$	$32 \times 72 =$	$82 \times 42 =$
$72 \times 62 =$	$22 \times 52 =$	$62 \times 92 =$
$32 \times 42 =$	$82 \times 32 =$	$72 \times 12 =$
$82^2 =$	$42^2 =$	$62^2 =$
$52^2 =$	$72^2 =$	$92^2 =$

Practice Problems II

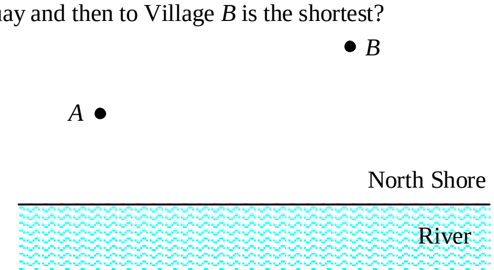
$23 \times 93 =$	$33 \times 73 =$	$83 \times 43 =$
$74 \times 64 =$	$24 \times 54 =$	$64 \times 94 =$
$75 \times 95 =$	$85 \times 35 =$	$96 \times 86 =$
$36 \times 56 =$	$57 \times 47 =$	$37 \times 87 =$
$28 \times 98 =$	$68 \times 58 =$	$79 \times 29 =$

Math Competition Skill

Where to Build a Quay?

Problem

There are two villages, marked by *A* and *B*, to the north of a river. The two villages are planning to build a quay on the north shore of the river. Where should the quay be built such that the total distance from Village *A* to the quay and then to Village *B* is the shortest?

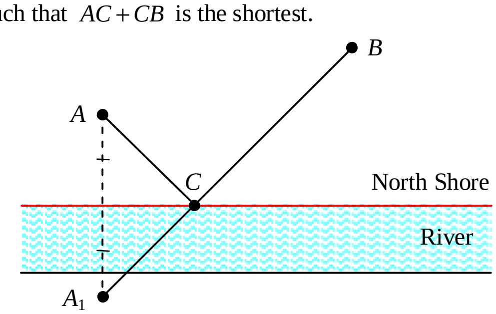


Solution

Suppose that the north shore is a straight line. Find the symmetrical point A_1 of *A* about the north shore.

Draw A_1B intersecting the north shore at *C*.

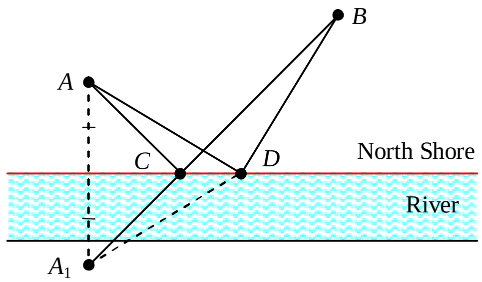
Point *C* is the location where the quay should be built such that $AC + CB$ is the shortest.



Why is $AC + CB$ shortest?

Let D be a point on the north shore different from C. We will prove that

$$AD + DB > AC + CB.$$



Draw A_1D.

Because of the symmetry, $AC = A_1C$ and $AD = A_1D$.

So $AC + CB = A_1C + CB$ and $AD + DB = A_1D + DB$.

By the construction of C, $A_1C + CB = A_1B$.

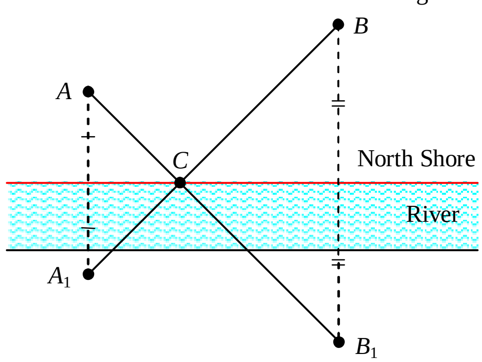
In ΔA_1BD, $A_1D + DB > A_1B$.

So $A_1D + DB > A_1C + CB$.

Therefore, $AD + DB > AC + CB$.

We have found the location for the quay starting from point A. Will the location for the quay be changed if we start from point B?

The answer is "no." This is shown in the figure below.

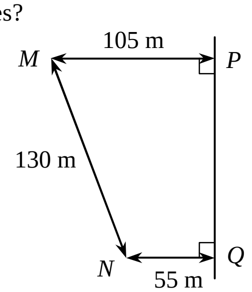


Gas Supply Lines

The problem is from Canadian Mathematics Competition (CMC).

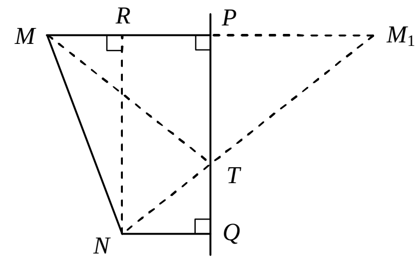
(CMC Grade 10 Cayley 1999 Problem 22)

A main gas line runs through P and Q. From some point T on PQ, a supply line runs to a house at point M. A second supply line from T runs to a house at point N. What is the minimum total length of pipe required for the two supply lines?



Answer: 200 m

Solution:



Extend MP to M_1 such that $PM_1 = PM = 105$ m.

Draw NM_1 intersecting PQ at T. Then T is the desired location.

Draw MT. We have $MT + TN = M_1T + TN = M_1N$ is the minimum total length of pipe.

Draw $NR \perp MP$ with R on MP. Then

$MR = 105 - 55 = 50$ m and $M_1R = 105 + 55 = 160$ m.

In right ΔMRN,

$$NR = \sqrt{MN^2 - MR^2} = \sqrt{130^2 - 50^2} = 120 \text{ m.}$$

In right ΔM_1RN,

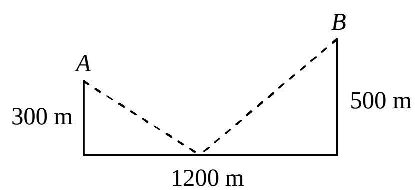
$$M_1N = \sqrt{M_1R^2 + NR^2} = \sqrt{160^2 + 120^2} = 200 \text{ m.}$$

Therefore, the minimum total length of the pipe is 200 m.

Practice Problems I

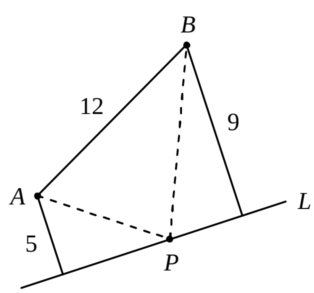
1. (MathCounts 2001 National Team Problem 4)

The rules for a race require that all runners start at A, touch any part of the 1200-meter wall, and stop at B. What is the minimum distance a participant must run?



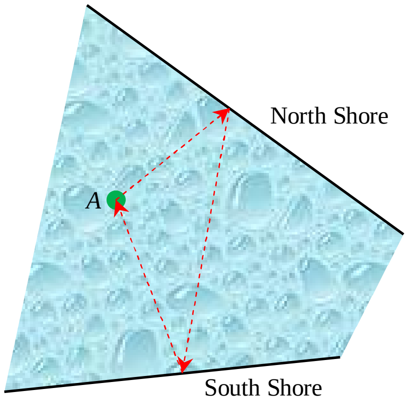
2. (University of North Colorado Mathematics Contest 2007-2008 Final Round Problem 6)

Points A and B are on the same side of line L in the plane. A is 5 units away from L, and B is 9 units away from L. The distance between A and B is 12. For all points P on L what is the smallest value of the sum $AP + PB$ of the distances from A to P and from P to B?

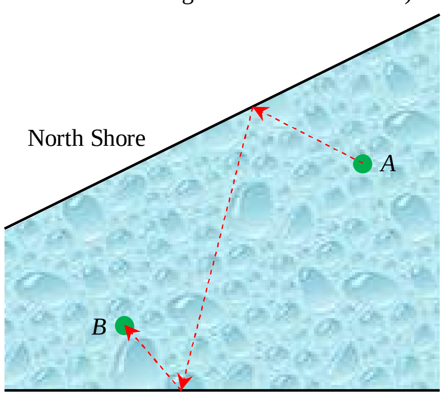


Practice Problems II

1. The figure shows a map of a lake with an island marked *A*. The rules for a race require that all competitors row a boat starting at island *A*, touching any part of the north shore, then touching any part of the south shore, and finally coming back to island *A*. Draw the route which has the shortest distance. (The route drawn in the figure is not the shortest.)



2. The figure shows a map of a lake with two islands marked *A* and *B*. The rules for a race require that all competitors row a boat starting at island *A*, touching any part of the north shore, then touching any part of the south shore, and finally stopping at island *B*. Draw the route which has the shortest distance. (The route drawn in the figure is not the shortest.)

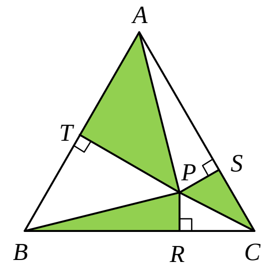


A Problem from a Real Math Competition

Today's Problem comes from Wisconsin Mathematics Science & Engineering Talent Search (WMSETS).

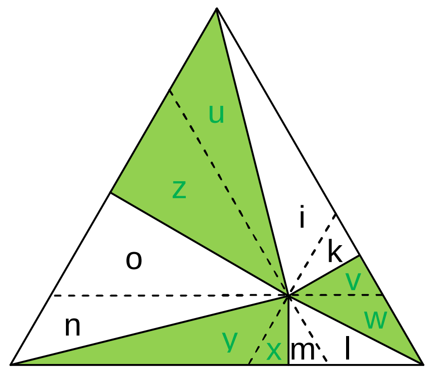
(WMSETS 2001-2002 Set II Problem 2)

In the figure, ΔABC is equilateral and *P* is some point in the interior of the triangle. Perpendiculars *PR*, *PS* and *PT* are dropped from *P* to the sides of the triangle, and lines are drawn from *P* to the vertices *A*, *B* and *C*. Prove that the sum of the areas of the three shaded triangles is exactly half of the area of ΔABC.



Proof:

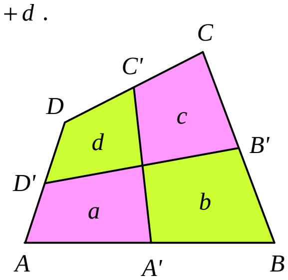
Through *P* draw three lines parallel to *AB*, *BC*, and *CA* respectively. We see the one-to-one correspondence between the shaded regions and the unshaded regions.



Practice Problem

(Norway Niels Henrik Abel Contest 1993 – Final Problem 1(a))

Let *ABCD* be a convex quadrilateral. Let A' be the midpoint of *AB*, B' be the midpoint of *BC*, C' be the midpoint of *CD*, and D' be the midpoint of *DA*. Draw $A'C'$ and $B'D'$, and let *a*, *b*, *c*, and *d* be the areas of the four minor quadrilaterals, as shown in the figure. Prove that $a+c=b+d$.



Answers to All Practice Problems in Last Issue

Math Trick: Mental Calculation

Practice Problems I

696	621	672
702	550	754

552	644	567
441	576	529
625	729	841
784	484	676

Practice Problems II

1248	1221	1292
2162	1890	2254
3009	3074	4692
5402	7310	8924

How Do You Count? – Surface Area of Cubes

1. $20\times6-24\times2=72$
2. $8\times24+12\times16-24\times2\times4=192$
3. $8\times6+12\times14-24\times2=168$

A Problem from a Real Math Competition

$36-9\pi$

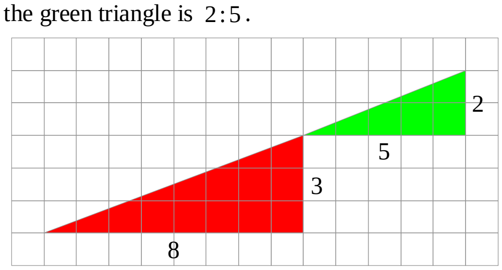
Solutions to Creative Thinking Problems 28 to 30

28. Cutting Birthday Cake

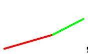
This is my solution:

29. Where does the Hole Come?

The red and green pieces are two triangles. The length ratio of the two legs in the red triangle is $3:8$, while that in the green triangle is $2:5$.

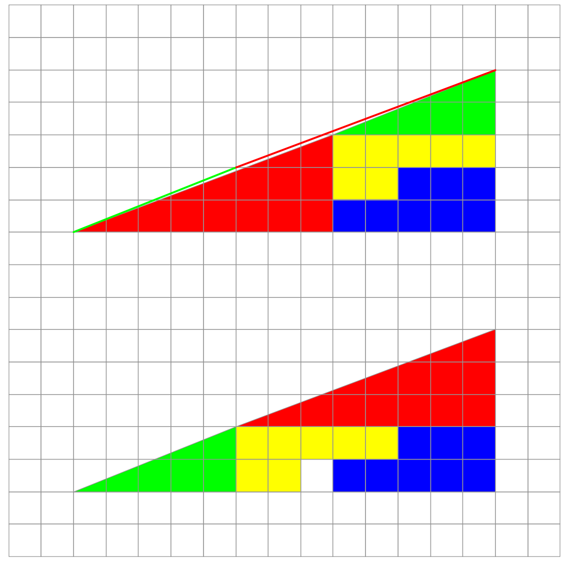


Since $3:8\neq2:5$, the two hypotenuses don't lie in a line.

Note that $\frac{3}{8}<\frac{2}{5}$. The two hypotenuses form a shape like , in the top figure while they form a shape like in the bottom figure.

If I place the four hypotenuses together in the real scale, we will see the figure below.

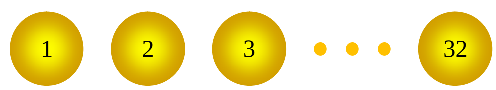
The four hypotenuses form a parallelogram. The area of the parallelogram is equal to 1, which is the area of the square hole in the bottom figure.



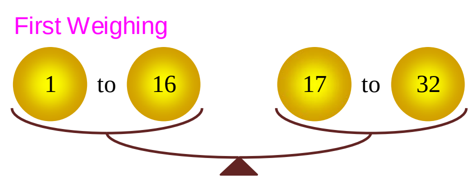
30. 32 Balls

In this problem we don't need to identify 2 fake balls. We just need to distribute 30 golden balls and 2 fake balls fairly to the two brothers.

Number the balls from 1 to 32.



Place Balls 1 to 16 on the left pan and Balls 17 to 32 on the right pan.



If the scale is in balance, we have completed dividing the balls as required. At this time each pan has 15 golden balls and one fake ball. We are done.

If the scale is not in balance, assume that the left side is heavier without loss of generality. The two fake balls are in one side.

Now we color the numbers 1 to 16 with red and the numbers 17 to 32 with green. A red number indicates that the corresponding ball is in the heavier side in the first weighing, while a green number means that the corresponding ball is in the lighter side.

We can conclude that

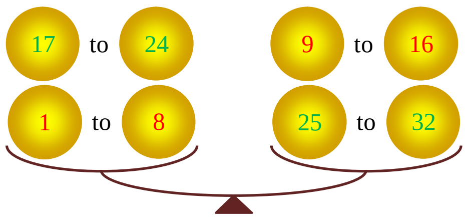
1. If a red ball is fake, it must be heavier than a real ball.
2. If a green ball is fake, it must be lighter than a real ball.

Now place Balls 1 to 8 and 17 to 24 on the left pan, and Balls 9 to 16 and 25 to 32 on the right pan.

If the scale is in balance, we are done.

If the scale is not, assume that the left side is heavier without loss of generality. The two fake balls are in one side.

Second Weighing



We conclude that Balls 9 to 16 and Balls 17 to 24 must be real.

Why?

Remember the meanings of the two colors. Now Balls 9 to 16 are in the lighter side. So none of them can be fake. Otherwise, this side must be heavier.

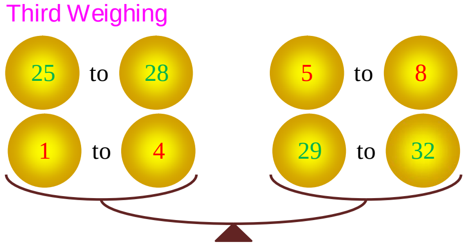
Similarly, Balls 17 to 24 cannot be fake.

Now we put Balls 9 to 16 and Balls 17 to 24 aside, and later distribute them equally to the two brothers.

Place Balls 1 to 4 and 25 to 28 on the left pan, and Balls 5 to 8 and 29 to 32 on the right pan.

If the scale is in balance, we are done.

Third Weighing

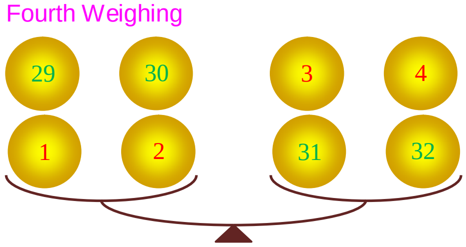


If it is not, assume that the left side is heavier without loss of generality. The two fake balls are in one side.

Similarly, Balls 5 to 8 and Balls 25 to 28 must be real. We put Balls 5 to 8 and Balls 25 to 28 aside.

Now we place Balls 1, 2, 29 and 30 on the left pan, and Balls 3, 4, 31 and 32 on the right pan.

Fourth Weighing



If the scale is in balance, we are done.

If it is not, assume the left side is heavier without loss of generality. The two fake balls are in one side.

Similarly, Balls 3 and 4 and Balls 29 and 30 must be real.

We conclude that the fake balls must be Balls 1 and 2, or Balls 31 and 32.

Now equally divide 28 real balls into two groups, and place Balls 1 and 31 into one group, and Balls 2 and 32 into the other group.

We are done.

I have to weigh four times to distribute 32 balls fairly.

Clues to Creative Thinking Problems 31 to 33

31. True or False

Assume a statement to be true one by one. Then examine whether there is any contradiction.

32. We Miss 2008

First try to obtain a number close to 2008.

33. MATH

Start from *M*. What must *M* be? Then what is the range of *A*?

Creative Thinking Problems 34 to 36

34. Make a Tent

Use 11 matchsticks to make a tent. Don't bend and overlap any matchsticks.

Your art work should be such that everybody may say "this is a tent" when he sees it.

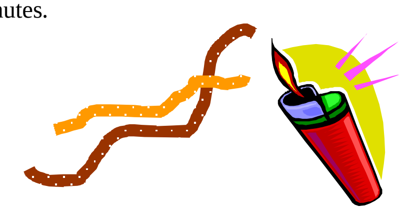
35. Filling 1 to 9

Fill digits 1 to 9 into the 9 squares such that the following expression is true. Each digit must be used once and only once.

□□ × □□ = □□ × □□□ = 4234

36. Two Ropes

Give you two ropes and a lighter. It is known that it takes one hour to burn out each rope from one end to the other. Use the two ropes and the lighter to measure a period of 45 minutes.



Note: The two ropes may not have the same length. The burning rate may not be uniform. That is, the burning of a rope may be sometimes faster and sometimes slower.

(Clues and solutions will be given in the next issues.)

