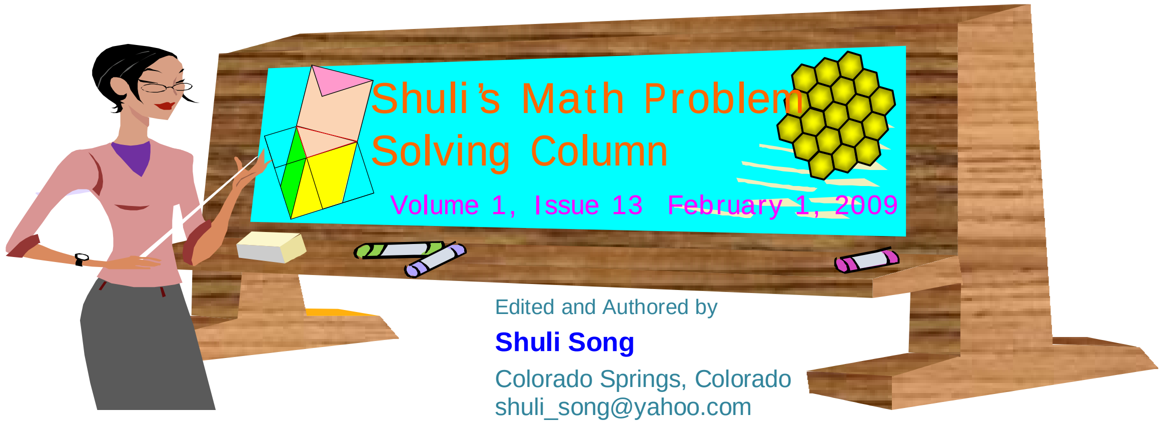

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Content

Math Trick

Mental Calculation: $\overline{20a}\times\overline{20b}$

The Trick

Mentally calculate:

$207\times 202=$ $\quad 206\times 205=$ $\quad 203\times 208=$

$206^2=$ $\quad 209\times 204=$ $\quad 204\times 207=$

Write these multiplications in the general form: $\overline{20a}\times\overline{20b}$ where *a* and *b* are digits.

A short cut to calculate the multiplications is shown through the following examples.

Example 1

Calculate 207×208.

Step 1: Calculate $\overline{20a}+b$.
In this example, $207+8=215$.

Step 2: Calculate $2\times(\overline{20a}+b)$.
In this example, $2\times 215=430$.

Step 3: Calculate $a\times b$.
In this example, $7\times 8=56$.

Step 4: Attach $a\times b$ as two digits to the right of the result in step 2. In this example, attach 56 to the right of 430: 43056.

Now we are done: $207\times 208=43056$.

Example 2

Calculate 202×204.

Step 1: Calculate $202+4=206$.

Step 2: Calculate $2\times 206=412$.

Step 3: Calculate $2\times 4=8$, treated as two digits: 08.

Step 4: Attach 08 to the right of 412: 41208.

We have $202\times 204=41208$.

Why Does This Work?

Write $\overline{20a}$ and $\overline{20b}$ in the base 10 representation:

$$\overline{20a}=200+a \text{ and } \overline{20b}=200+b.$$

So we have

$$\begin{aligned}&\overline{20a}\times\overline{20b}\\&=(200+a)\times(200+b)=40000+200b+200a+ab\\&=200(200+a+b)+ab=200(\overline{20a}+b)+ab.\end{aligned}$$

This shows that to calculate $\overline{20a}\times\overline{20b}$, we may do

Step 1: Calculate $\overline{20a}+b$.

Step 2: Multiply the result in step 1 by 2.

Step 3: Calculate $a\times b$.

Step 4: Attaching $a\times b$ as two digits to the right of $2\times(\overline{20a}+b)$.

Mental Calculation: $\overline{n0a}\times\overline{n0b}$

The similar procedure applies to the multiplications in the form $\overline{n0a}\times\overline{n0b}$ where *n* is a digit greater than 2. Instead of 2 we multiply $\overline{n0a}+b$ by *n* in step 2.

Example 3

Calculate 304×307.

Step 1: Calculate $304+7=311$.

Step 2: Calculate $3\times 311=933$.

Step 3: Calculate $4\times 7=28$.

Step 4: Attach 28 to the right of 933: 93328.

We have $304\times 307=93328$.

Example 4

Calculate 405×409.

Step 1: Calculate $405+9=414$.

Step 2: Calculate $4\times 414=1656$.

Step 3: Calculate $5\times 9=45$.

Step 4: Attach 45 to the right of 1656: 165645.

We obtain $405\times 409=165645$.

Example 5

Calculate 807×805.

Step 1: Calculate $807+5=812$.

Step 2: Calculate $8\times 812=6496$.

Step 3: Calculate $7\times 5=35$.

Step 4: Attach 35 to the right of 6496: 649635.

Then $807\times 805=649635$.

Practice Problems I

$202\times 209=$	$203\times 207=$	$208\times 204=$
$207\times 206=$	$202\times 205=$	$206\times 209=$
$201\times 209=$	$208\times 203=$	$209\times 208=$
$203\times 204=$	$206\times 205=$	$207\times 202=$
$208^2=$	$207^2=$	$206^2=$

Practice Problems II

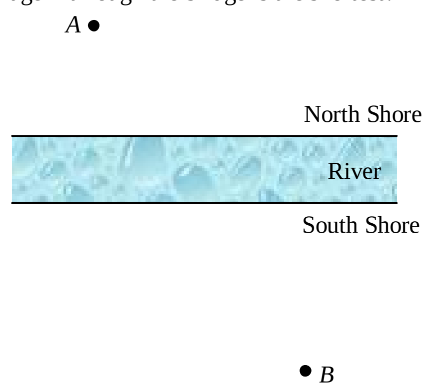
$307\times 306=$	$309^2=$	$308\times 305=$
$404\times 409=$	$408\times 403=$	$409\times 408=$
$503\times 504=$	$506\times 505=$	$507^2=$
$608^2=$	$604\times 606=$	$703\times 708=$
$702^2=$	$806\times 803=$	$906^2=$

Math Competition Skill

Where to Build a Bridge?

Problem

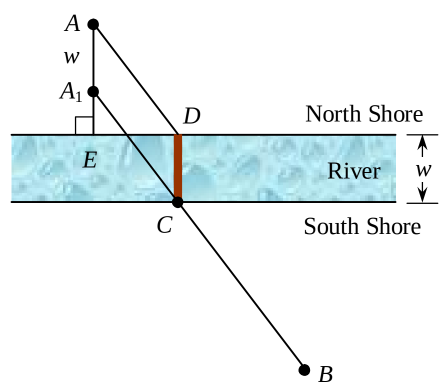
There are two villages located on the two sides of a river respectively. The villages are marked with *A* and *B*. The two villages are planning to build a bridge across the river, which is perpendicular to the shores that are considered as two parallel straight lines. Where should the bridge be built such that the distance from Village *A* to Village *B* through the bridge is the shortest?



Solution

Let *w* be the width of the river. Draw AE perpendicular to the north shore with *E* on the north shore. Locate A_1 on *AE* or its extension such that $AA_1=w$.

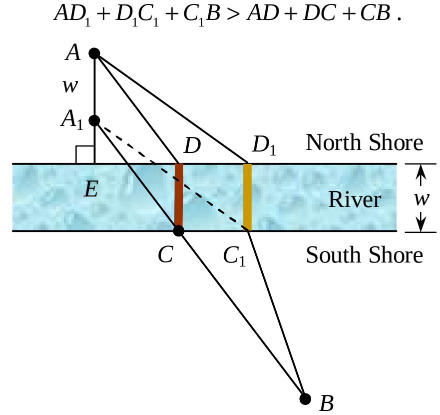
Draw A_1B, intersecting the south shore at *C*. Draw *CD* perpendicular to the shores, intersecting the north shore at *D*. Then *CD* is the location where the bridge should be built such that $AD+DC+CB$ is the shortest.



Why is $AD+DC+CB$ shortest?

Let C_1D_1 be a location different from *CD* for the bridge. We will prove that

$$AD_1+D_1C_1+C_1B>AD+DC+CB.$$



Draw A_1C_1. Since $AA_1 \parallel CD$ (perpendicular to the shores) and $AA_1 = CD$, AA_1CD is a parallelogram. So $AD = A_1C$.

Similarly $AA_1C_1D_1$ is a parallelogram. So $AD_1 = A_1C_1$.

Then we have

$$AD_1 + C_1B = A_1C_1 + C_1B \text{ and } AD + CB = A_1C + CB.$$

By the construction of *C*, $A_1C + CB = A_1B$.

In ΔA_1C_1B, $A_1C_1 + C_1B > A_1B$.

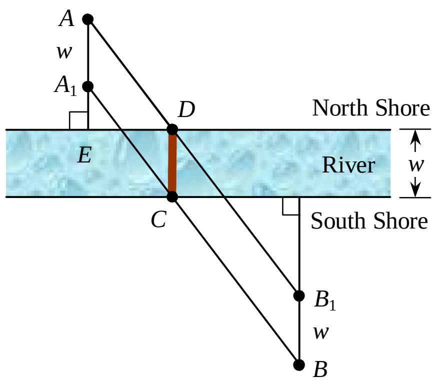
So $AD_1 + C_1B > AD + CB$.

Since $C_1D_1 = CD$, we obtain

$$AD_1 + D_1C_1 + C_1B > AD + DC + CB.$$

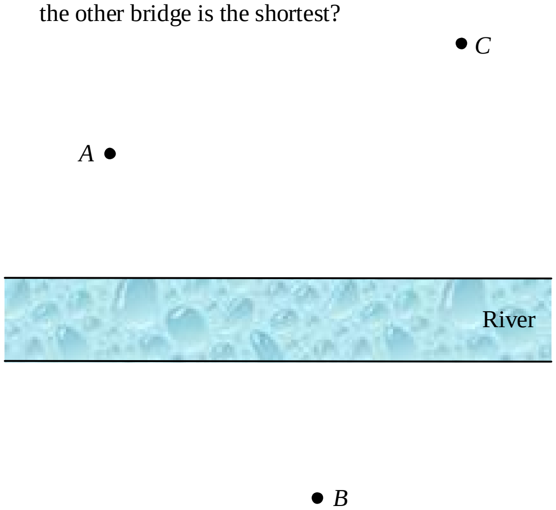
We have found the location for the bridge starting from Point *A*. Will the location where the bridge is to be built be changed if we start from Point *B*?

The answer is "no." This is shown in the figure below.

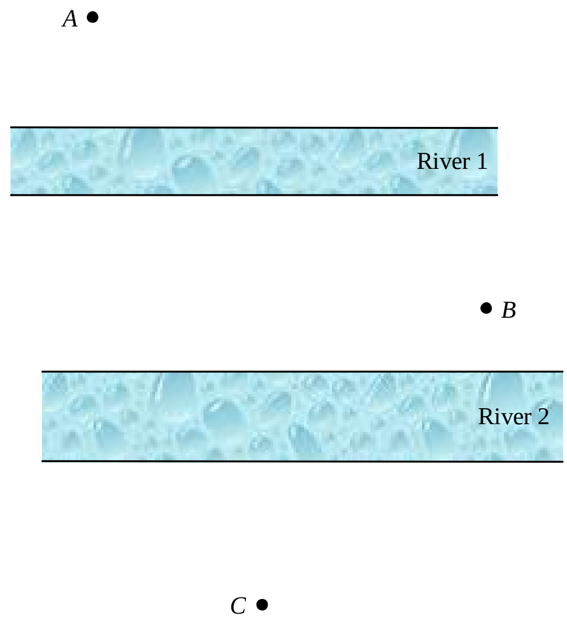


Practice Problems

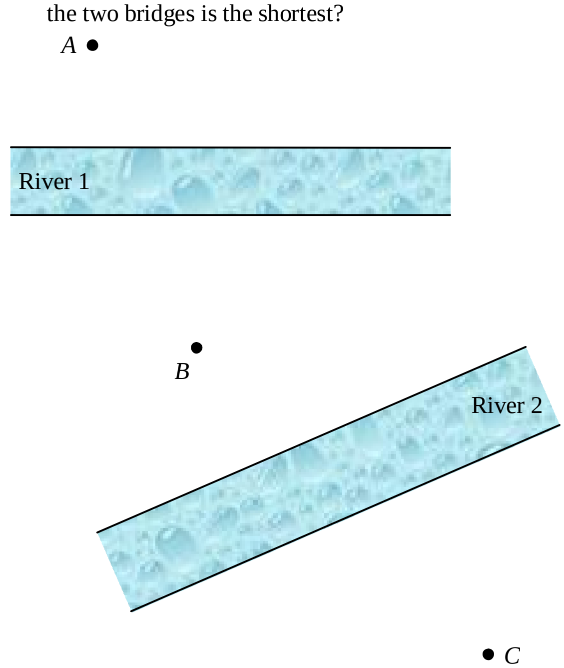
1. The map below shows a river and three villages *A*, *B*, and *C*. The villages are planning to build two bridges across the river. The shores of the river are two parallel lines. The bridges must be perpendicular to the shores. Where should the two bridges be built such that the total distance from Village *A* to Village *B* through one bridge and then to Village *C* through the other bridge is the shortest?



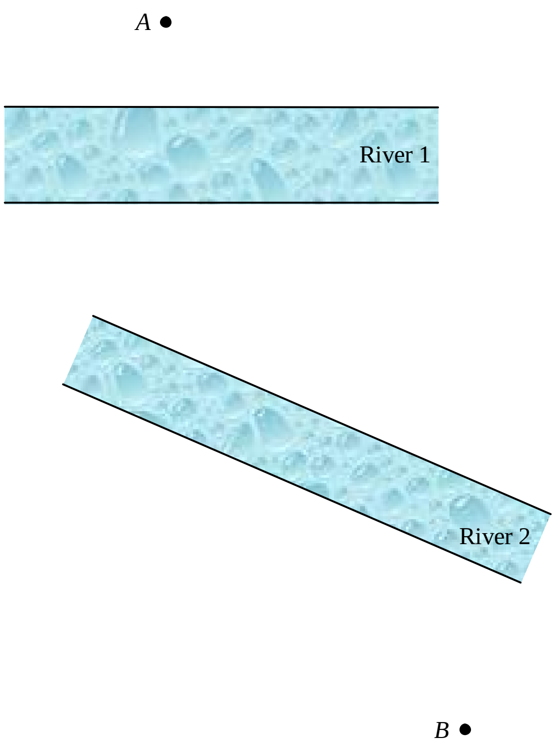
2. The map below shows two rivers and three villages *A*, *B*, and *C*. The villages are planning to build two bridges across the two rivers respectively. The shores of each river are two parallel lines. The bridges must be perpendicular to the shores. Where should the two bridges be built such that the total distance from Village *A* to Village *B* and then to Village *C* through the two bridges is the shortest?



3. The map below shows two rivers and three villages *A*, *B*, and *C*. The villages are planning to build two bridges across the two rivers respectively. The shores of each river are two parallel lines. The bridges must be perpendicular to the shores. Where should the two bridges be built such that the total distance from Village *A* to Village *B* and then to Village *C* through the two bridges is the shortest?



4. The map below shows two rivers and two villages *A* and *B*. The villages are planning to build two bridges across the two rivers respectively. The shores of each river are two parallel lines. The bridges must be perpendicular to the shores. Where should the two bridges be built such that the distance from Village *A* to Village *B* through the two bridges is the shortest?



A Problem from a Real Math Competition

Today's problem comes from Calgary Junior Math Contest (CJMC).

(CJMC 2007 Problem B5)

In Alberta, a 6% tax is added to the cost of all purchases. If an item costs x dollars, the tax is computed by calculating $0.06x$, rounded to the nearest cent (with half cents rounded up). A price is called impossible if it cannot be the price of an item after tax is added.

(a) Prove that \$9.98 is an impossible price.

(b) How many impossible prices are there less than or equal to \$10.00? That is, how many of the prices

1¢, 2¢, $\ldots$, 99¢, \$1.00, \$1.01, $\ldots$, \$9.99, \$10.00

are impossible?

Proof of (a)

Calculate:

$9.41+0.06\times 9.41=9.9746$ rounded to 9.97,

$9.42+0.06\times 9.42=9.9852$ rounded to 9.99.

So \$9.98 is an impossible price.

Answer to (b): 57

Solution One to (b):

1¢ is possible since $0.01+0.06\times 0.01=0.0106$ rounded to 0.01.

Similarly, 2¢, 3¢, 4¢, 5¢, 6¢, 7¢, and 8¢ are all possible.

9¢ is impossible because $0.08+0.06\times 0.08=0.0848$ rounded to 0.08, and $0.09+0.06\times 0.09=0.0954$ rounded to 0.10.

If we study further, the second impossible price is 26¢, the third impossible price is 44¢, etc. If we list all impossible prices, we have the following sequence:

9, 26, 44, 62, 79, 97, 115, 132, 150, 168, 185, $\ldots$

Starting at 9, we repeatedly add 17, 18, and 18 to the previous number to obtain the next number.

Note that $17+18+18=53$.

Since $9+19\times 53=1016$, \$10.16 is the first impossible price over \$10.

So there are $3\times 19=57$ impossible prices which are less than or equal to \$10.

Solution one is not so elegant, in which we have focused on prices after tax. In solution two we will obtain the answer by considering prices before tax.

Solution Two to (b):

Note that $\frac{10.00}{1.06}=9.433\ldots$.

Since $9.43+9.43\times 0.06=9.9958$ rounded to 10, \$9.43 is the highest price before tax such that the price after tax is less than or equal to \$10.

So 1¢ + tax, 2¢ + tax, 3¢ + tax, $\ldots$, \$9.43 + tax are all possible prices. There are 943 possible prices.

Therefore, there are $1000-943=57$ impossible prices.

Practice Problem

In Colorado, a 7.4% tax is added to the cost of all purchases. If an item costs x dollars, the tax is computed by calculating $0.074x$, rounded to the nearest cent (with half cents rounded up). A price is called impossible if it cannot be the price of an item after tax is added. How many impossible prices are there less than or equal to \$100.00?

Answers to All Practice Problems in Last Issue

Math Trick: Mental Calculation

Practice Problems I

3864	2304	3444
4464	1144	5704
1344	2624	864
6724	1764	3844
2704	5184	8464

Practice Problems II

2139	2409	3569
4736	1296	6016
7125	2975	8256
2016	2679	3219
2744	3944	2291

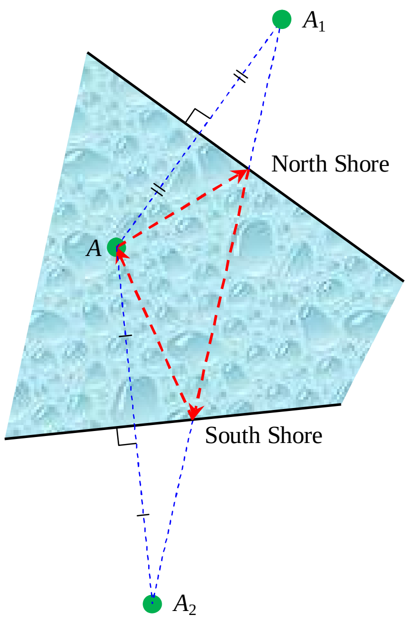
Where to Build a Quay?

Practice Problems I

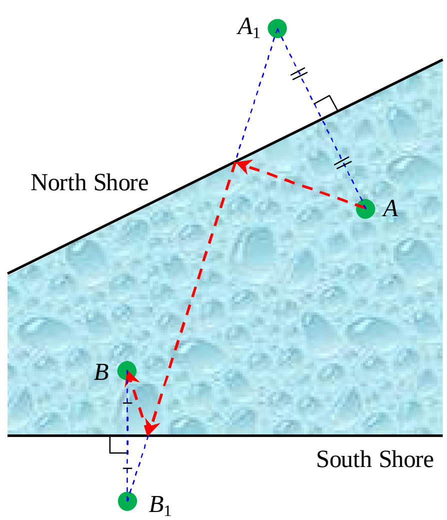
1. $400\sqrt{13}$
2. 18

Practice Problems II

1. The route is drawn.



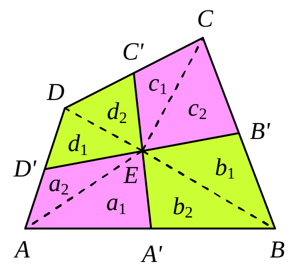
2. The route is drawn.



A Problem from a Real Math Competition

Let $A'C'$ and $B'D'$ intersect at *E*.

Draw *AE*, *BE*, *CE*, and *DE*.



Then a is divided into a_1 and a_2, b is divided into b_1 and b_2, c is divided into c_1 and c_2, and d is divided into d_1 and d_2. That is,

$$a = a_1 + a_2,\ b = b_1 + b_2,\ c = c_1 + c_2, \text{ and } d = d_1 + d_2.$$

Look at ΔABE. A' is the midpoint of *AB*. So

$$a_1 = b_2 .$$

Similarly,

$$b_1 = c_2,\ c_1 = d_2, \text{ and } d_1 = a_2 .$$

Therefore,

$$\begin{aligned} a + c &= a_1 + a_2 + c_1 + c_2 = b_2 + d_1 + d_2 + b_1 \\ &= b_1 + b_2 + d_1 + d_2 = b + d. \end{aligned}$$

Solutions to Creative Thinking Problems 31 to 33

31. True or False

Assume that a statement is true one by one, and examine whether we obtain any contradictions.

The first statement "all of the following are true" must be false because the problem says "exactly one of the five statements is true."

Similarly the fourth statement "all of the above are true" must be false.

Assume that the third statement "some of the following are true" is true. Then at least one of the two following statements is true. We must have at least two true statements. This contradicts to that exactly one of the five statements is true.

Assume that the fifth statement "none of the above is true" is true. This means that the third statement "some of the following are true." is false. Then none of the following statements is true. This implies that the fifth statement is false. It is a contradiction to the assumption that the fifth statement is true.

There must be one true statement. So the second statement must be true.

In fact, if we assume that the second statement is true, we will not get any contradictions.

32. We Miss 2008

Starting from $345 \times 6 = 2070$, I have the following two solutions:

$$1+2+3 \quad 4 \quad 5\times 6-7\times 8-9=2008,$$

$$1+2+3 \quad 4 \quad 5\times 6+7-8\times 9=2008.$$

33. MATH

First $T=0$.

We should start with the 9, the only number given in the expression.

Since $MATH \times M = 9\square\square\square$, M is 3.

Then A is a digit not greater than 3 because there is no carry in $A \times M$. Note that different letters represent different digits. Since $T=0$ and $M=3$, A is 1 or 2.

Since $MATH \times H$ gives four digits, H is also a digit not greater than 3. So H is 1 or 2.

Now we see two cases: $A=1$, $H=2$ or $A=2$, $H=1$.

In the first case, the multiplication is

				3	1	0	2
			×	3	1	0	2
				6	2	0	4
		3	1	0	2		
	9	3	0	6			
□	9	6	2	2	4	0	4

There are only 8 digits in the product.

Then the second case must work:

				3	2	0	1
			×	3	2	0	1
				3	2	0	1
		6	4	0	2		
	9	6	0	3			
1	0	2	4	6	4	0	1

Therefore, MATH is 3201.

Clues to Creative Thinking Problems 34 to 36

34. Make a Tent

This is a tricky problem.

35. Filling 1 to 9

Factor.

36. Two Ropes

$30+15=45$. Burning a rope from both ends lasts 30 minutes.

Creative Thinking Problems 37 to 39

37. True or False Again

The following statements are read on a piece of paper:

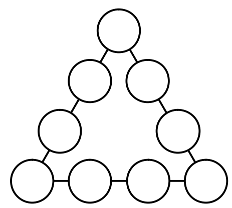
1. There is no more than one true statement.
2. There are no more than two true statements.
3. There are no more than three true statements.
4. There are no more than four true statements.
5. There are no more than five true statements.

Which statements are true?

38. Another Magic Triangle

After you have solved Creative Thinking Problem 25, the following problem becomes easy.

Fill 1 through 9 into 9 circles with a number in each circle such that the sum of four numbers on each side of the triangle is the same.



The same sum is called the *magic number*.

What is the smallest possible magic number, and what is the possible greatest magic number.

39. 10 × 10 Matrix

A 10 by 10 matrix is given with the numbers 0 to 99 filled a shown. Place 50 "+" and 50 "–" signs, one before each number, such that each row and each column has exactly 5 +'s and 5 –'s.

What is the highest possible total sum of the 100 numbers (after signs are placed), and what is the lowest possible?

0	1	2	3	4	5	6	7	8	9
10	11	12	13	14	15	16	17	18	19
20	21	22	23	24	25	26	27	28	29
30	31	32	33	34	35	36	37	38	39
40	41	42	43	44	45	46	47	48	49
50	51	52	53	54	55	56	57	58	59
60	61	62	63	64	65	66	67	68	69
70	71	72	73	74	75	76	77	78	79
80	81	82	83	84	85	86	87	88	89
90	91	92	93	94	95	96	97	98	99

(Clues and solutions will be given in the next issues.)