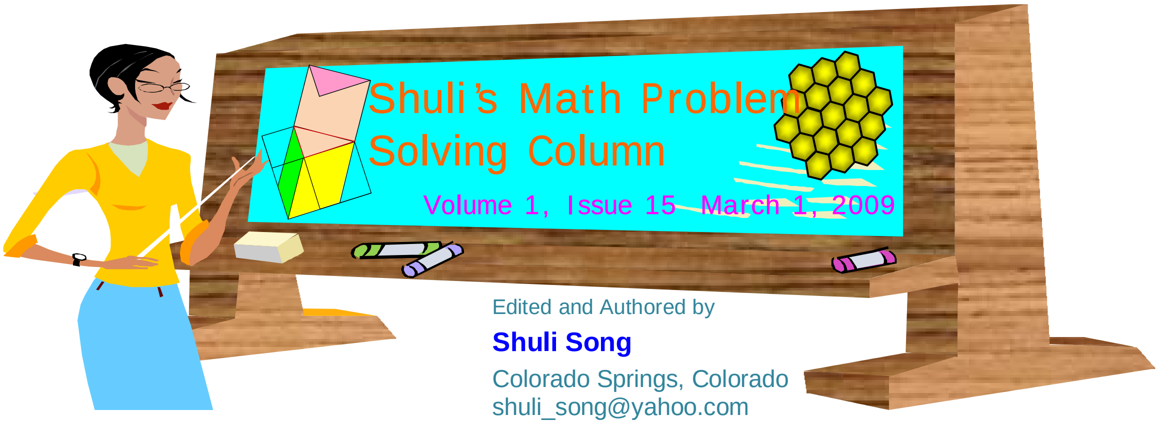

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Content

Math Trick

Mental Calculation: $\overline{aa} \times \overline{bc}$

The Trick

This short lesson will teach how to obtain the products for the following multiplications mentally.

$77 \times 28 =$	$88 \times 64 =$	$33 \times 55 =$
$99 \times 73 =$	$66 \times 19 =$	$44 \times 37 =$

There are two properties in common for all these pairs of two-digit numbers:

1. One number has two digits which are the same.
2. One number has two digits which sum to 10.

We write the multiplications in the general form: $\overline{aa} \times \overline{bc}$ where *a*, *b*, and *c* are digits with $b + c = 10$.

Following the given steps you will be able to complete one multiplication in 3 seconds.

Example 1

Calculate 77×46.

Step 1: Calculate $a \times (b+1)$.
In this example, $7 \times (4+1) = 35$.

Step 2: Calculate $a \times c$.
In this example, $7 \times 6 = 42$.

Step 3: Attach the result in step 2 as two digits to the right of the result in step 1.
In this example, attach 42 to the right of 35: 3542.

Now we are done: $77 \times 46 = 3542$.

Example 2

Calculate 44×82.

Step 1: Calculate $4 \times (8+1) = 36$.

Step 2: Calculate $4 \times 2 = 8$, treated as two digits: 08

Step 3: Attach 08 to the right of 36: 3608.

We have $44 \times 82 = 3608$.

Why Does This Work?

Write $\overline{aa}$ and $\overline{bc}$ in the base 10 representation:

$$\overline{aa} = 10a + a \text{ and } \overline{bc} = 10b + c.$$

So we have

$$\overline{aa} \times \overline{bc} = (10a + a) \times (10b + c)$$
$$= 100ab + 10ab + 10ac + ac = 100ab + 10a(b+c) + ac$$

Note that $b + c = 10$. We obtain

$$\overline{aa} \times \overline{bc} = 100ab + 100a + ac = 100a(b+1) + ac$$

This shows that to calculate $\overline{aa} \times \overline{bc}$, we may do

Step 1: Calculate $a \times (b+1)$.

Step 2: Calculate $a \times c$.

Step 3: Attach $a \times c$ as two digits to the right of $a \times (b+1)$.

Practice Problems

$66 \times 37 =$	$99 \times 91 =$	$33 \times 64 =$
$22 \times 73 =$	$55 \times 46 =$	$77 \times 37 =$
$44 \times 82 =$	$11 \times 37 =$	$66 \times 19 =$
$64 \times 88 =$	$55 \times 66 =$	$91 \times 44 =$
$82 \times 99 =$	$28 \times 44 =$	$73 \times 88 =$

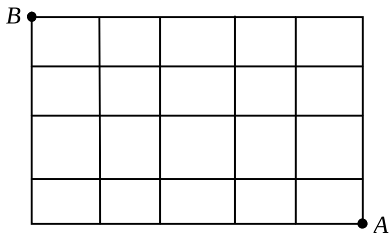
Math Competition Skill

Routes in $m \times n$ Grid City

Problem

The following problem is popular in math competitions.

The figure below shows the road map of 4×5 Grid City. How many shortest routes are there from A to B?



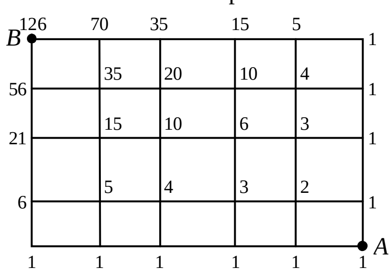
The problem may be presented in the general form:

Find the number of shortest routes from one corner to the opposite corner in $m \times n$ Grid City.

Solution

Solution One:

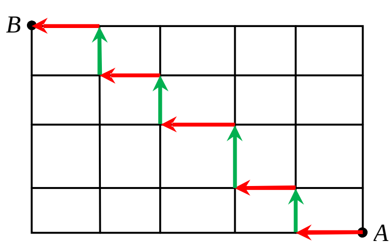
We are equipped with a tool called "*Making Numbers*" from the last issue to solve this problem.

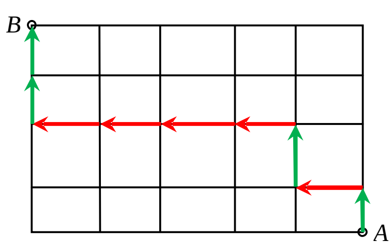


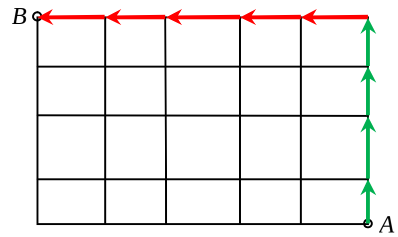
There are 126 shortest routes.

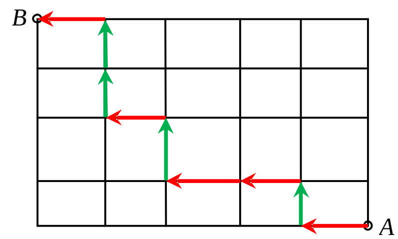
Solution Two:

We draw several shortest routes:









In any shortest route we must go 5 horizontal blocks (marked with red) and 4 vertical blocks (marked with green).

Use "*H*" to represent a horizontal block and "*V*" a vertical block. Any route corresponds to a 9-letter string of 5 *H*'s and 4 *V*'s in some order. The above illustrated routes correspond to the four strings: "*HVHVHVHVH*", "*VHVHHHHVV*", "*VVVVHHHHH*", "*HVHHVHVVH*" respectively. Given a 9-letter string of 5 *H*'s and 4 *V*'s there is a shortest route.

So the number of strings is equal to the number of shortest routes.

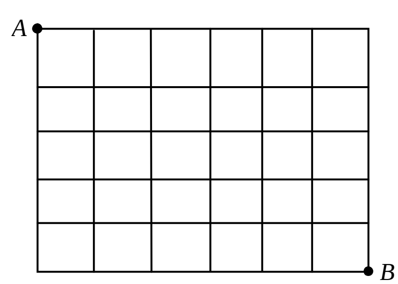
There are $\binom{9}{4}$ 9-letter strings consisting of 5 *H*'s and 4 *V*'s. Therefore, there are $\binom{9}{4} = 126$ shortest routes.

In general, in $m \times n$ Grid City we must go n blocks horizontally and m blocks vertically. There are $\binom{m+n}{m}$ $(m+n)$-letter strings consisting of n *H*'s and m *V*'s.

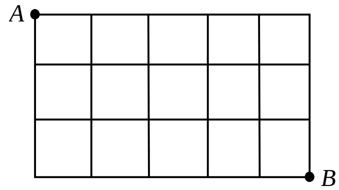
So $\binom{m+n}{m}$ or $\binom{m+n}{n}$ is the general formula for the number of shortest routes.

Practice Problems

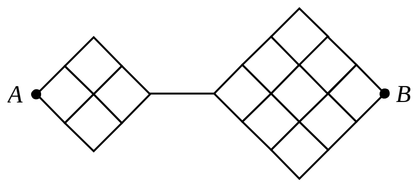
1. The figure below shows the road map of 5×6 Grid City. How many shortest routes are there from A to B?



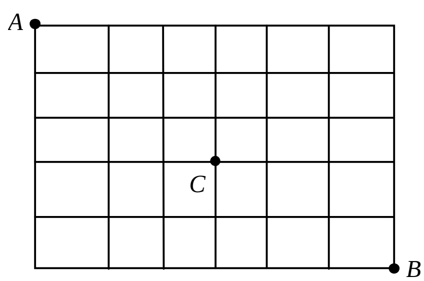
2. The figure below shows the road map of 3×5 Grid City. How many shortest routes are there from *A* to *B*?



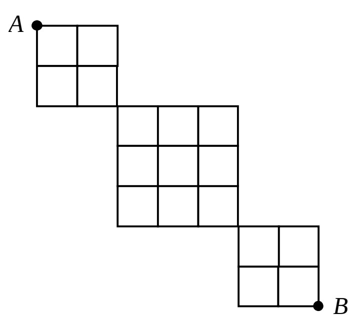
3. The figure below shows the road map of Grid City. How many shortest routes are there from *A* to *B*?



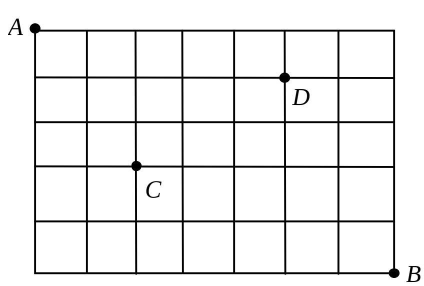
4. The figure below shows the road map of Grid City. How many shortest routes are there from *A* to *B* if we must pass through *C*?



5. The figure below shows the road map of Grid City. How many shortest routes are there from *A* to *B*?



6. The figure below shows the road map of Grid City. How many shortest routes are there from *A* to *B* if we must pass through *C* and *D*?

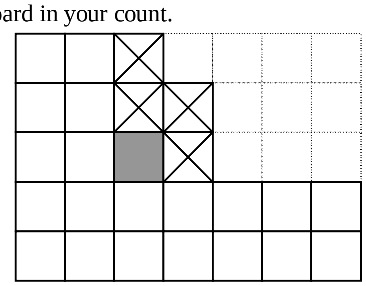


A Problem from a Real Math Competition

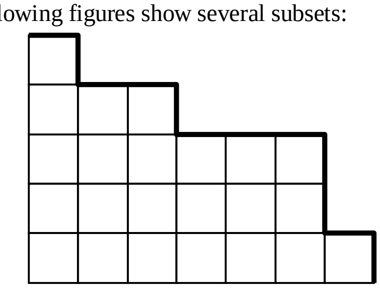
Today's problem comes from American Invitational Mathematics Examination (AIME).

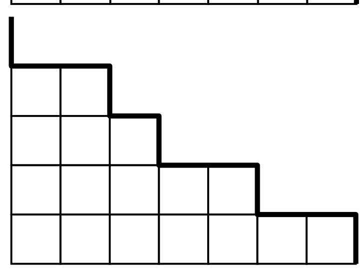
(10th AIME 1992 Problem 12)

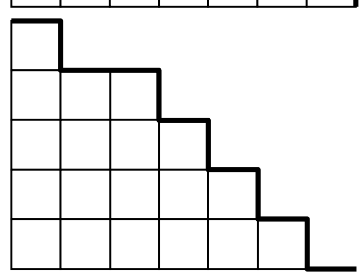
In a game of *Chomp*, two players alternatively take "bites" from a 5-by-7 grid of unit squares. To take a bite, the player chooses one of the remaining squares, then removes ("eats") all squares found in the quadrant defined by the left edge (extended upward) and lower edge (extended rightward) of the chosen square. For example, the bite determined by the shaded square in the diagram would remove the shaded square and the four squares marked by x. (The squares with two more dotted edges have been removed from the original board in previous moves.) The object of the game is to make one's opponent take the last bite. The diagram shows one of the many subsets of the set of 35 unit squares that can occur during games of Chomp. How many different subsets are there in all? Include the full board and the empty board in your count.

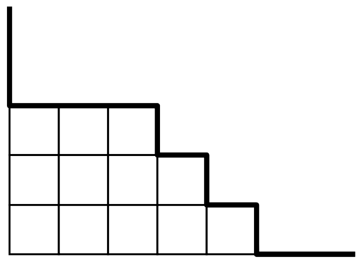


The following figures show several subsets:



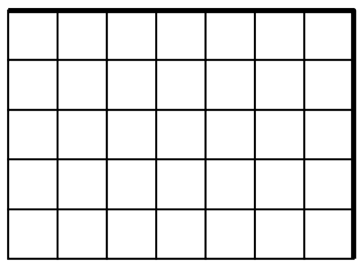






I have marked the topmost and rightmost edges of the unit squares in each subset.

We observe that one subset corresponds to a shortest route along the grid lines from the right-bottom corner to the left-top corner. The full board and the empty board correspond to the following two routes respectively.



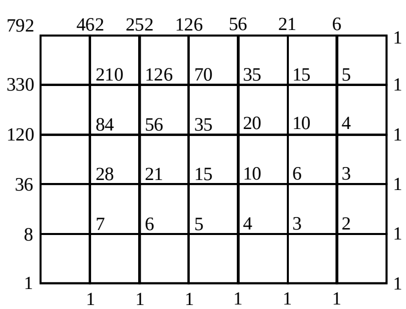
So the problem is exactly the same as the following problem:

How many shortest routes are there in 5×7 Grid City from the right-bottom corner to the left-top corner?

Answer: 792

Solution One (Marking Numbers):

We mark numbers as shown. The answer is 792.

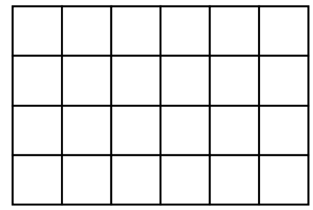


Solution Two (Use the Formula):

The answer is $\binom{5+7}{5}=\binom{12}{5}=\frac{12\cdot11\cdot10\cdot9\cdot8}{5\cdot4\cdot3\cdot2\cdot1}=792$.

Practice Problem

In the game of *Chomp*, how many different subsets of squares are there if the grid is 4×6?



Answers to All Practice Problems in Last Issue

Math Trick: Mental Calculation

Practice Problems I

384	8036	4899
4896	891	2484
6375	1591	3596
4864	1575	899
6336	2451	8084

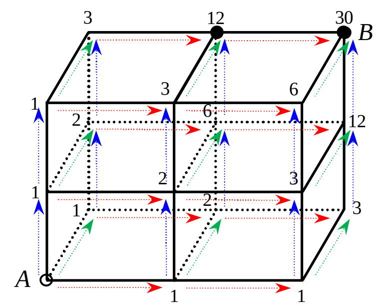
Practice Problems II

14384	9991	19575
159991	62464	12019
1439991	1102464	72836
1322475	4409919	9548096

All Roads to Rome

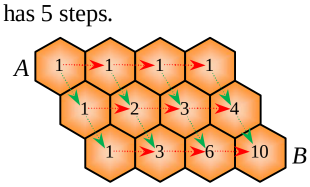
Practice Problems I

1. 70
2. 38
3. 10
4. 13
5. 61
6. 90
7. 30. We mark numbers as shown below.



Practice Problems II

1. 6
2. 144
3. 60
4. 91
5. 110
6. 10. You must go directly right or down-right so that the path has 5 steps.



A Problem from a Real Math Competition

1. *Solution One:*

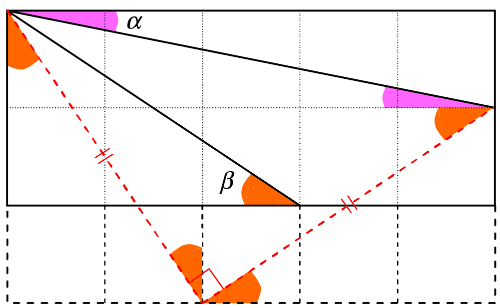
Note that $\tan\alpha = \frac{1}{5}$ and $\tan\beta = \frac{2}{3}$.

So $\tan(\alpha+\beta) = \frac{\tan\alpha + \tan\beta}{1-\tan\alpha\tan\beta} = \frac{\frac{1}{5}+\frac{2}{3}}{1-\frac{1}{5}\cdot\frac{2}{3}} = 1$.

Thus $\alpha + \beta = 45°$

Solution Two:

Attach another layer of five squares. The angles equal to α are marked with pink, and the angles equal to β are marked with orange. We see an isosceles right triangle. One pink angle and one orange angle make $45°$.



2. *Solution One:*

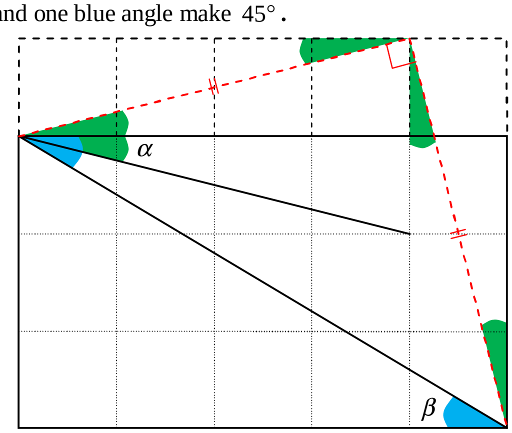
We know that $\tan\alpha = \frac{1}{4}$ and $\tan\beta = \frac{3}{5}$.

So $\tan(\alpha+\beta) = \frac{\frac{1}{4}+\frac{3}{5}}{1-\frac{1}{4}\cdot\frac{3}{5}} = 1$.

Then $\alpha + \beta = 45°$

Solution Two:

Attach another layer of five squares. The angles equal to α are marked with green, and the angles equal to β are marked with blue. From the isosceles right triangle formed, we know that one green angle and one blue angle make $45°$.



3. *Solution One:*

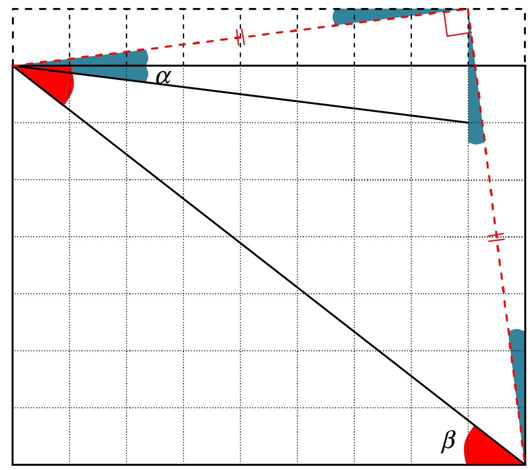
Note that $\tan\alpha = \frac{1}{8}$ and $\tan\beta = \frac{7}{9}$.

So $\tan(\alpha+\beta) = \frac{\frac{1}{8}+\frac{7}{9}}{1-\frac{1}{8}\cdot\frac{7}{9}} = 1$.

We have $\alpha + \beta = 45°$

Attach another layer of nine squares. The angles equal to α are marked with blue, and the angles equal to β are marked with red. We observe an isosceles right triangle. One blue angle and one red angle make $45°$.

Solution Two:



4. *Solution One:*

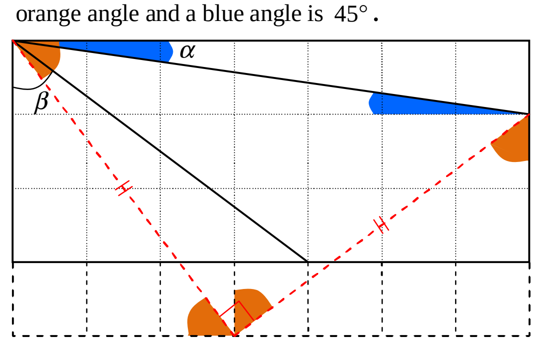
We know that $\tan\alpha = \frac{1}{7}$ and $\tan\beta = \frac{4}{3}$.

So $\tan(\beta-\alpha) = \frac{\tan\beta - \tan\alpha}{1+\tan\beta\tan\alpha} = \frac{\frac{4}{3}-\frac{1}{7}}{1+\frac{4}{3}\cdot\frac{1}{7}} = 1$.

We obtain $\beta - \alpha = 45°$

Solution Two:

Attach another layer of seven squares. The angles equal to α are marked with blue, and the angles equal to β are marked with orange. The isosceles right triangle shows that the difference between an orange angle and a blue angle is $45°$.



5. *Solution One:*

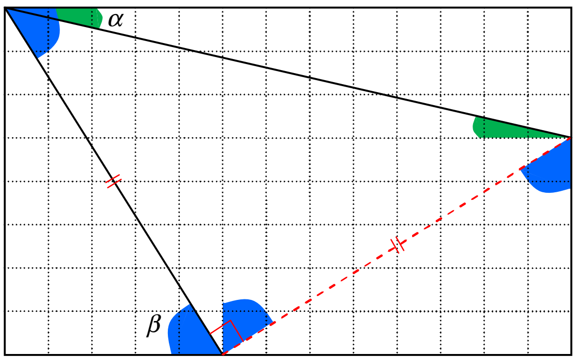
Note that $\tan\alpha = \frac{3}{13}$ and $\tan\beta = \frac{8}{5}$.

So $\tan(\beta-\alpha) = \dfrac{\frac{8}{5}-\frac{3}{13}}{1+\frac{8}{5}\cdot\frac{3}{13}} = 1$.

Thus $\beta - \alpha = 45°$

Solution Two:

The angles equal to β are marked with blue, and the angles equal to α are marked with green. The isosceles right triangle shows that the difference between a blue angle and a green angle is $45°$.



Solutions to Creative Thinking Problems 37 to 39

37. True or False Again

Assume that statement 1 is true. That is, there is no more than one true statement. Then statements 2, 3, 4, and 5 must be also true. Now we have more than one true statement. It is a contradiction.

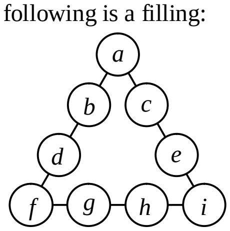
Similarly, the assumption that statement 2 is true will lead to a contradiction.

Assume that statement 3 is true. That is, there are no more than three true statements. Then statements 4 and 5 must be also true. Now we have three true statements. No contradiction can be found.

Therefore, statements 3, 4, and 5 are true.

38. Another Magic Triangle

Assume that the following is a filling:



where *a, b, c, d, e, f, g, h,* and *i* are 1 to 9 in some order. Then $a+b+c+d+e+f+g+h+i=45$.

Let M be the magic number. That is,

$a+b+d+f=M$, $f+g+h+i=M$, $i+e+c+a=M$.

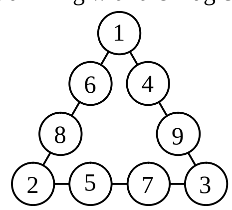
Adding them we have $a+f+i=3(M-15)$.

It follows that the sum of the three numbers filled at the vertices must be divisible by 3.

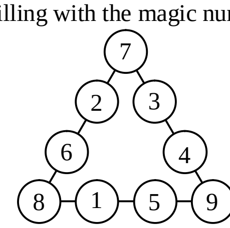
Rewriting the above expression we have the magic number $M = 15 + \frac{a+f+i}{3}$.

In order to get the smallest possible magic number we have to make $a+f+i$ as small as possible. So we assign 1, 2, and 3 for *a, f,* and *i* respectively. The smallest magic number is $M = 15 + (1+2+3)/3 = 17$.

The following is a filling with the magic number 17:



In order to get the greatest possible magic number we have to make $a+f+i$ as great as possible. So we assign 7, 8, and 9 for *a, f,* and *i* respectively. The greatest magic number is $M = 15 + (7+8+9)/3 = 23$. The following is a filling with the magic number 23:



39. 10 × 10 Matrix

If we cannot see the answer immediately, we may do some experiments.

For example, we assign a "+" for each number in the left top 5×5 matrix and the right bottom 5×5 matrix, and a "–" for each number in the left bottom 5×5 matrix and the right top 5×5 matrix. This arrangement satisfies the given conditions.

+0	+1	+2	+3	+4	–5	–6	–7	–8	–9
+10	+11	+12	+13	+14	–15	–16	–17	–18	–19
+20	+21	+22	+23	+24	–25	–26	–27	–28	–29
+30	+31	+32	+33	+34	–35	–36	–37	–38	–39
+40	+41	+42	+43	+44	–45	–46	–47	–48	–49
–50	–51	–52	–53	–54	+55	+56	+57	+58	+59
–60	–61	–62	–63	–64	+65	+66	+67	+68	+69
–70	–71	–72	–73	–74	+75	+76	+77	+78	+79
–80	–81	–82	–83	–84	+85	+86	+87	+88	+89
–90	–91	–92	–93	–94	+95	+96	+97	+98	+99

The sum of the 10 numbers in each of five rows from the first to the fifth is –25, and the sum of the 10 numbers in each of five rows from the sixth to the tenth is 25.

Therefore, the total sum of all 100 numbers is 0.

What do we think now? Is it always true or did it happen by chance that the total sum is 0?

Let us do another experiment.

This time we will place "+" and "–" as shown.

+0	–1	+2	–3	+4	–5	+6	–7	+8	–9
–10	+11	–12	+13	–14	+15	–16	+17	–18	+19
+20	–21	+22	–23	+24	–25	+26	–27	+28	–29
–30	+31	–32	+33	–34	+35	–36	+37	–38	+39
+40	–41	+42	–43	+44	–45	+46	–47	+48	–49
–50	+51	–52	+53	–54	+55	–56	+57	–58	+59
+60	–61	+62	–63	+64	–65	+66	–67	+68	–69
–70	+71	–72	+73	–74	+75	–76	+77	–78	
+80	–81	+82	–83	+84	–85	+86	–87	+88	–89
–90	+91	–92	+93	–94	+95	–96	+97	–98	+99

This arrangement satisfies the given conditions.

In this filling the sum of the 10 numbers in the *i*th row is -5 for $i = 1, 3, 5, 7, 9$, and +5 for $i = 2, 4, 6, 8, 10$.

Therefore, the total sum of all 100 numbers is 0.

Now we are quite confident that the sum is always 0.

However, we have to prove this.

It is not a proof until we have tried all possible fillings.

My proof is presented as follows:

I place a zero at the tens position of a number if that number has only one digit.

00	01	02	03	04	05	06	07	08	09
10	11	12	13	14	15	16	17	18	19
20	21	22	23	24	25	26	27	28	29
30	31	32	33	34	35	36	37	38	39
40	41	42	43	44	45	46	47	48	49
50	51	52	53	54	55	56	57	58	59
60	61	62	63	64	65	66	67	68	69
70	71	72	73	74	75	76	77	78	79
80	81	82	83	84	85	86	87	88	89
90	91	92	93	94	95	96	97	98	99

To add and subtract we may separate the tens and ones digits.

For example, to calculate $25 + 78 - 47$ we may calculate $20 + 70 - 40 = 50$ for the tens digits and $5 + 8 - 7 = 6$ for the ones digits, then put them together:

$$25 + 78 - 47 = 50 + 6 = 56.$$

For this problem we may consider the tens and ones digits separately.

Separate the original matrix into the following two matrices:

Tens Matrix

00	00	00	00	00	00	00	00	00	00
10	10	10	10	10	10	10	10	10	10
20	20	20	20	20	20	20	20	20	20
30	30	30	30	30	30	30	30	30	30
40	40	40	40	40	40	40	40	40	40
50	50	50	50	50	50	50	50	50	50
60	60	60	60	60	60	60	60	60	60
70	70	70	70	70	70	70	70	70	70
80	80	80	80	80	80	80	80	80	80
90	90	90	90	90	90	90	90	90	90

Ones Matrix:

0	1	2	3	4	5	6	7	8	9
0	1	2	3	4	5	6	7	8	9
0	1	2	3	4	5	6	7	8	9
0	1	2	3	4	5	6	7	8	9
0	1	2	3	4	5	6	7	8	9
0	1	2	3	4	5	6	7	8	9
0	1	2	3	4	5	6	7	8	9
0	1	2	3	4	5	6	7	8	9
0	1	2	3	4	5	6	7	8	9
0	1	2	3	4	5	6	7	8	9

Since there are five "+"and five "–" in each row, the sum in each of the 10 rows is 0 in the tens matrix. So the total sum of all 100 numbers in the tens matrix is 0. It doesn't depend on how you put the "+" and "–" signs satisfying the given conditions.

Since there are five "+"and five "–" in each column. The sum in each of the 10 columns is 0 in the ones matrix. So the total sum of all 100 numbers in the ones matrix is also 0. It doesn't depend on how you put the "+" and "–" either.

Therefore, the total sum of all 100 numbers in the original matrix is $0 + 0 = 0$ no matter how you put the "+" and "–" signs satisfying the given conditions.

Now let us summarize the answer to the problem.

The possible highest total sum is 0, and the possible lowest total sum is also 0.

Clues to Creative Thinking Problems 40 to 42

40. The Average of 4 and 6 is not 5!

Assume a number for the distance.
It is better to assume a common multiple of 4 and 6 for the distance.

41. Making 24 with 1, 5, 5, and 5

$5 \times 4.8 = 24$.

42. Aging Faster?

It seems that one year has passed from yesterday to the next year. But the girl becomes 3 years older. We have to find where the 2 extra years comes from.

Notice that when a girl is 13 years 364 days old, she is still 13 years old. Also notice that January 1 is in the next year, but December 31 is also in the next year.

Creative Thinking Problems 43 to 45

43. Liar, Truth-teller, and Neutral

The inhabitants in an island can be categorized into three types:

Liars, who always lie.

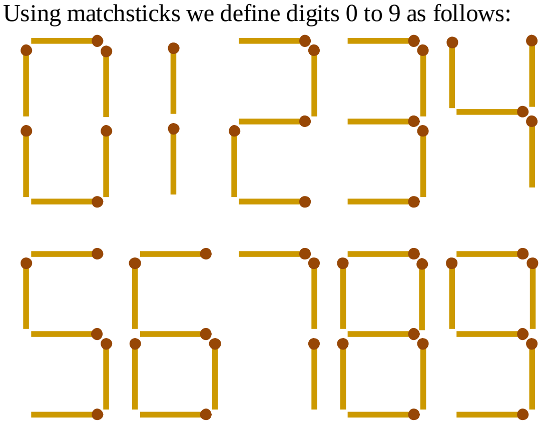
Truth-tellers, who always tell the truth.

Neutrals, who sometimes lie and sometimes tell the truth.

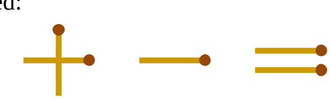
One day you met one inhabitant, and he told you "I'm not a truth-teller". What type of inhabitants was he? Or is it impossible to determine?

44. 6 + 8 = 3

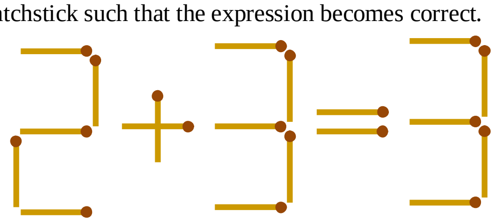
Using matchsticks we define digits 0 to 9 as follows:

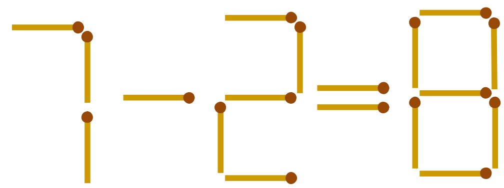


The three mathematical symbols "+","–", and "=" are also defined:

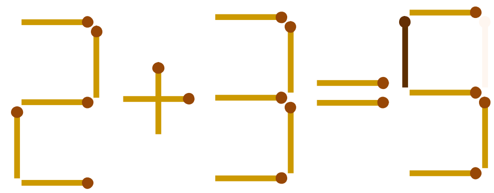


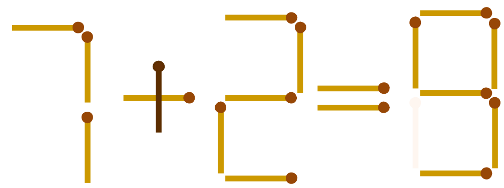
For each of the following two expressions move one matchstick such that the expression becomes correct.



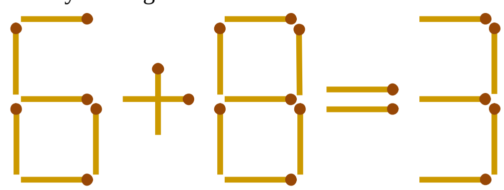


For these two we can readily obtain the solutions:

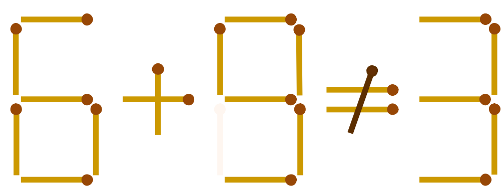




Find two solutions to make the following expression correct by moving one matchstick:



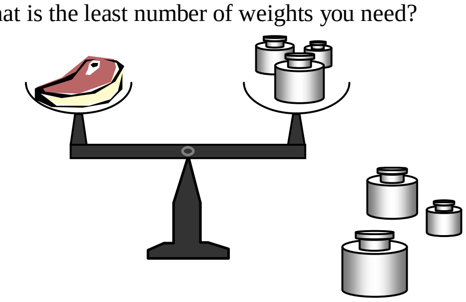
Since the inequality symbol is not defined, the following solution is not allowed:



45. Weighing Meat I

You have a pan scale to weigh meat up to 100 pounds. Any piece of meat has a weight of whole pounds, that is, 1 pound, 2 pounds, 3 pounds etc. In the weighing, you must place meat on the left pan and weight(s) on the right pan.

What is the least number of weights you need?



(Clues and solutions will be given in the next issues.)