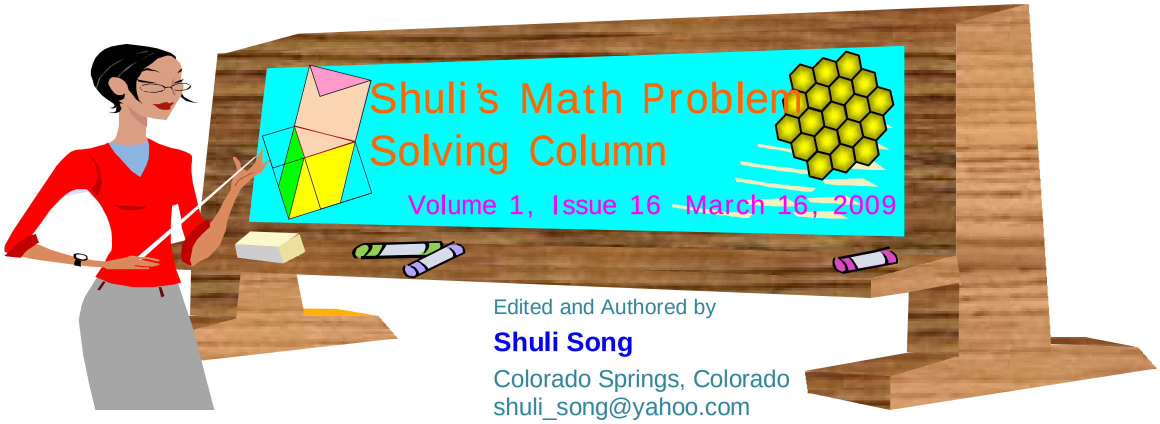

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Content

Math Trick

Mental Calculation: $\overline{9a}\times\overline{9b}$

The Trick

Mentally calculate:

$97\times98=$	$96\times93=$	$95\times92=$
$99\times94=$	$91\times92=$	$98\times95=$

To calculate the product of two numbers close to 100, we have a short cut.

Write the multiplications in the general form: $\overline{9a}\times\overline{9b}$ where a and b are digits. Let $c=100-\overline{9a}$ and $d=100-\overline{9b}$. Then $\overline{9a}\times\overline{9b}=(100-c)\times(100-d)$.

The steps are shown through the following examples.

Example 1

Calculate 93×98.

Step 1: Calculate $c=100-\overline{9a}$ and $d=100-\overline{9b}$.
In this example, $c=100-93=7$ and $d=100-98=2$.

Step 2: Calculate $\overline{9a}-d$ or $\overline{9b}-c$.
In this example, $\overline{9a}-d=93-2=91$ or $\overline{9b}-c=98-7=91$.

Step 3: Calculate $c\times d$.
In this example, $7\times2=14$.

Step 4: Attach the result in step 3 as two digits to the right of the result in step 2.
In this example, attach 14 to the right of 91: 9114.

Now we are done: $93\times98=9114$.

Example 2

Calculate 98×97.

Step 1: Calculate $100-98=2$ and $100-97=3$.

Step 2: Calculate $98-3=95$ or $97-2=95$.

Step 3: Calculate $2\times3=6$, treated as two digits: 06

Step 4: Attach 06 to the right of 95: 9506.

We have $98\times97=9506$.

This works for any two numbers close to 100.

Example 3

Calculate 89×93.

Step 1: Calculate $100-89=11$ and $100-93=7$.

Step 2: Calculate $89-7=82$ or $93-11=82$.

Step 3: Calculate $11\times7=77$.

Step 4: Attach 77 to the right of 82: 8277.

We have $89\times93=8277$.

Example 4

Calculate 86×89.

Step 1: Calculate $100-86=14$ and $100-89=11$.

Step 2: Calculate $86-11=75$ or $89-14=75$.

Step 3: Calculate $14\times11=154$.

Step 4: Add 1, the hundreds digit of 154, to 75 yielding 76, and attach 54 to the right of 76: 7654.

We have $86\times89=7654$.

Why Does This Work?

$$\overline{9a}\times\overline{9b} = (100-c)\times(100-d) = 10000-100c-100d+cd$$
$$= 100(100-c-d)+cd = 100(\overline{9a}-d)+cd$$
or
$$= 100(\overline{9b}-c)+cd$$

This shows that to calculate $\overline{9a}\times\overline{9b}$, we may do

Step 1: Calculate $c=100-\overline{9a}$ and $d=100-\overline{9b}$.

Step 2: Calculate $\overline{9a}-d$ or $\overline{9b}-c$.

Step 3: Calculate $c\times d$.

Step 4: Attach $c\times d$ as two digits to the right of $\overline{9a}-d$ or $\overline{9b}-c$.

Practice Problems I

$96\times97=$	$91\times98=$	$93\times95=$
$91\times96=$	$95\times96=$	$99\times92=$
$92\times94=$	$94\times98=$	$97\times92=$
$98\times93=$	$93\times99=$	$97\times95=$
$97^2=$	$96^2=$	$94^2=$

Practice Problems II

$86\times97=$	$91\times87=$	$93\times85=$
$81\times96=$	$95\times86=$	$89\times92=$
$87\times84=$	$86\times88=$	$87\times82=$
$88\times83=$	$89\times85=$	$84\times83=$
$87^2=$	$86^2=$	$84^2=$

Math Competition Skill

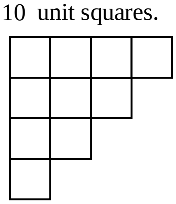
How Do You Count? – Rectangles in Tableaus

Problem

This is an interesting problem from the 2005-2006 University of Northern Colorado Mathematics Contest (UNCMC).

(UNCMC 2005-2006 Final Round Problem 11)

Call the figure below a 4-tableau shape, in which we have $4+3+2+1=10$ unit squares.



Find the number of rectangles of all sizes contained in this shape. Note that a square is a rectangle.

Generate a formula for the number of rectangles of all sizes contained in an *n*-tableau shape, in which we have

$$n+(n-1)+\cdots+2+1=\frac{n(n+1)}{2}$$

unit squares.

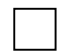
Solution

For the 4-tableau shape, we can obtain the answer by systematically listing.

Let $f(n)$ be the number of rectangles in an *n*-tableau shape.

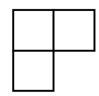
Let us study for $n=1, 2, 3, 4, \cdots$.

$n=1$:



Obviously, $f(1)=1$.

$n=2$:

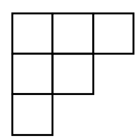


We systematically list in the following table.

Shape	Number of Rectangles
1×1	$2+1=3$
1×2 or 2×1	$2\times1=2$
Total	5

So $f(2)=5$.

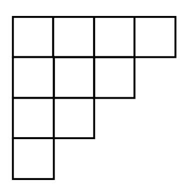
$n=3$



Shape	Number of Rectangles
1×1	$3+2+1=6$
1×2 or 2×1	$2\times(2+1)=6$
1×3 or 3×1	$2\times1=2$
2×2	1
Total	15

Then $f(3)=15$.

$n=4$:



Shape	Number of Rectangles
1×1	$4+3=2+1=10$
1×2 or 2×1	$2\times(3+2+1)=12$
1×3 or 3×1	$2\times(2+1)=6$

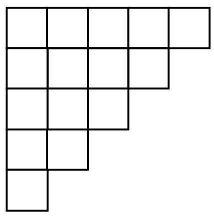
1×4 or 4×1	$2\times1=2$
2×2	$1+2=3$
2×3 or 3×2	$2\times1=2$
Total	35

So $f(4)=35$.

So the answer to the first part of the problem is 35.

Let us study further.

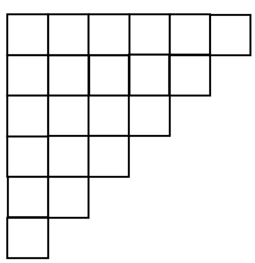
$n=5$:



Shape	Number of Rectangles
1×1	$5+4+3+2+1=15$
1×2 or 2×1	$2\times(4+3+2+1)=20$
1×3 or 3×1	$2\times(3+2+1)=12$
1×4 or 4×1	$2\times(2+1)=6$
1×5 or 5×1	$2\times1=2$
2×2	$3+2+1=6$
2×3 or 3×2	$2\times(2+1)=6$
2×4 or 4×2	$2\times1=2$
3×3	1
Total	70

We have $f(5)=70$.

$n=6$:



Shape	Number of Rectangles
1×1	$6+5+4+3+2+1=21$
1×2 or 2×1	$2\times(5+4+3+2+1)=30$
1×3 or 3×1	$2\times(4+3+2+1)=20$
1×4 or 4×1	$2\times(3+2+1)=12$
1×5 or 5×1	$2\times(2+1)=6$
1×6 or 6×1	$2\times1=2$
2×2	$4+3+2+1=10$
2×3 or 3×2	$2\times(3+2+1)=12$
2×4 or 4×2	$2\times(2+1)=6$

2×5 or 5×2	$2\times1=2$
3×3	$2+1=3$
3×4 or 4×3	$2\times1=2$
Total	126

Then $f(6)=126$.

We obtain the sequence: 1, 5, 15, 35, 70, 126, $\cdots$.

The differences form the sequence 4, 10, 20, 35, 56, $\cdots$, which we call the difference sequence of the 1st order.

The differences in differences form the sequence 6, 10, 15, 21, $\cdots$, which we call the difference sequence of the 2nd order.

The difference sequence of the 3rd order is 4, 5, 6, $\cdots$.

The difference sequence of the 4th order is 1, 1, $\cdots$, which are constant.

So the general formula is a polynomial of degree 4. We may determine the polynomial with the *Method of Undetermined Coefficients*.

Let me write the formula:

$$f(n)=\binom{n+3}{4}.$$

Now we check:

$$f(1)=\binom{4}{4}=1;$$

$$f(2)=\binom{5}{4}=\binom{5}{1}=5;$$

$$f(3)=\binom{6}{4}=\binom{6}{2}=\frac{6\times5}{2\times1}=15;$$

$$f(4)=\binom{7}{4}=\binom{7}{3}=\frac{7\times6\times5}{3\times2\times1}=35;$$

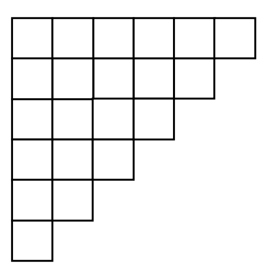
$$f(5)=\binom{8}{4}=\frac{8\times7\times6\times5}{4\times3\times2\times1}=70;$$

$$f(6)=\binom{9}{4}=\frac{9\times8\times7\times6}{4\times3\times2\times1}=126.$$

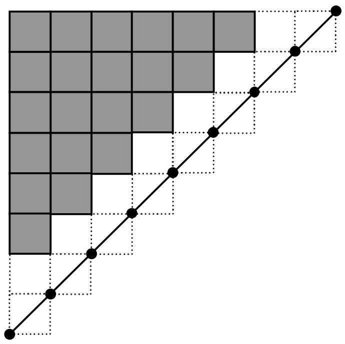
All are correct.

Why Do We Have This Formula?

Consider a 6-tableau. We will derive the general formula by a mathematical model in which a one-to-one correspondence is established.

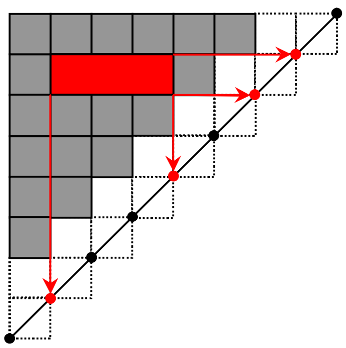


We attach $7+8=15$ squares as shown. Nine points are marked. Note that $9=6+3$.

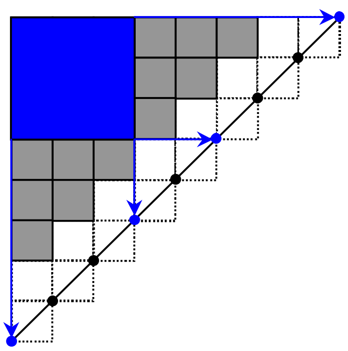


We claim that there is a one-to-one correspondence between the rectangles in the 6-tableau and the selections of four points from the nine points.

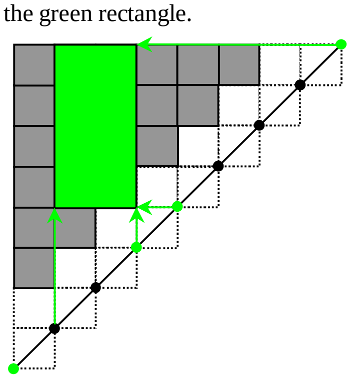
For example, the red rectangle corresponds to the four red points.



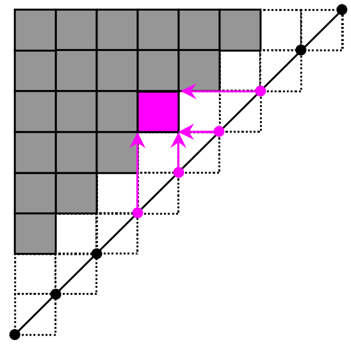
The blue rectangle corresponds to the four blue points.



The reverse is also true. The four green points uniquely determine the green rectangle.



The four pink points uniquely determine the pink rectangle.

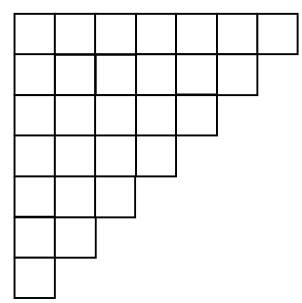


So the number of rectangles in a 6-tableau is $\binom{9}{4}$.

In general the number of rectangles in an *n*-tableau is $f(n)=\binom{n+3}{4}$.

Practice Problem

Practice systematically listing by finding $f(7)$. Use the given table.



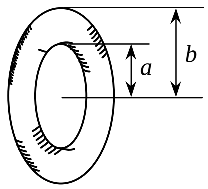
Shape	Number of Rectangles
1×1	
1×2 or 2×1	
1×3 or 3×1	
1×4 or 4×1	
1×5 or 5×1	
1×6 or 6×1	
1×7 or 7×1	
2×2	
2×3 or 3×2	
2×4 or 4×2	
2×5 or 5×2	
2×6 or 6×2	
3×3	
3×4 or 4×3	
3×5 or 5×3	
4×4	
Total	

A Problem from a Real Math Competition

Today's problem comes from Canadian Mathematics Competition (CMC).

(CMC 1984 Grade 11 Fermat Problem 24)

A formula for the volume of the doughnut with inner radius *a* and outer radius *b*, as shown, is



(A) $\pi\left(\frac{a+b}{2}\right)^2$ (B) $\frac{1}{3}\pi^2(a+b)^3$

(C) $\frac{1}{4}\pi^2(a+b)(b-a)^2$ (D) $\pi^3(b^2-a^2)$

(E) $\frac{1}{3}\pi(b^3+a^3)$

Answer: C

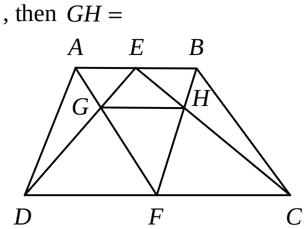
Solution:

It may be difficult to derive a formula for the volume of a doughnut. However, we know that the volume is the product of three dimensions. So A and D cannot be the answer. If *a* is very close to *b*, the volume should be very close to 0. In choices B, C, and E, only C possesses this property.

Therefore, the answer is C.

Practice Problem

$ABCD$ is a trapezoid with $AB \parallel CD$. E and F are the midpoints of AB and CD respectively. AF and DE intersect at *G*, and *BF* and *CE* intersect at *H*. If $AB = a$ and $CD = b$, then $GH =$



(A) $\frac{a+b}{4}$ (B) $b-a$ (C) $\frac{b-a}{2}$

(D) $\sqrt{ab}$ (E) $\frac{ab}{a+b}$

Answers to All Practice Problems in Last Issue

Math Trick: Mental Calculation

2442	9009	2112
1606	2530	2849
3608	407	1254
5632	3630	4004
8118	1232	6424

Routes in *m* × *n* Grid City

1. 462	2. 56	3. 120
4. 200	5. 720	6. 1500

A Problem from a Real Math Competition

210

Solutions to Creative Thinking Problems 40 to 42

40. The Average of 4 and 6 is not 5!

Assume that the distance in one way is *s* km.

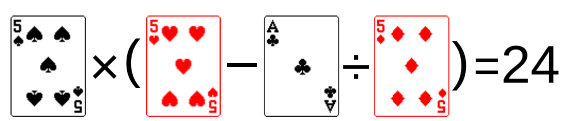
Then the time required for going up the mountain is $\frac{s}{4}$ hr., and the time required for coming down is $\frac{s}{6}$ hr.

So the total time for the whole trip is $\frac{s}{4}+\frac{s}{6}=\frac{5}{12}s$ hr.

The total distance is $2s$ km. Thus the average speed for the whole trip is $\frac{2s}{\frac{5}{12}s}=\frac{24}{5}=4.8$ km/hr., which is not equal to 5 km/hr.

Note that we can simplify the process by assuming that the distance in one way is 12 km.

41. Making 24 with 1, 5, 5, and 5



42. Aging Faster?

It seems that one year has passed from yesterday to the next year. But the girl becomes 3 years older. Where does the 2 extra years come from?

If we put "today" as January 1, there are almost two years to December 31 of the next year. This allows us to make up for one year.

Even though a girl is 13.99 L , she is still 13 years old. This allows us to make up for another year.

The following is my solution:

"Today" is January 1, and her birthday is December 31.

The day before yesterday was December 30.

When I ask the girl "how old are you?" today, she answers "I was 13 on the day before yesterday." Actually, she was $13\frac{364}{365}$ (or $13\frac{365}{366}$ depending on whether the leap year it was) on December 30.

Yesterday (December 31 of the last year) was her 14th birthday. She will turn 15 on December 31 this year. She will be 16 years old next year.

Clues to Creative Thinking Problems 43 to 45

43. Liar, Truth-teller, and Neutral

When you ask a liar and a truth-teller "are you a liar?", both of them will reply "no."

When you ask a liar and a truth-teller "are you a truth-teller?", both of them will reply "yes."

This is always the key to solve a "liar and truth-teller" problem.

44. 6 + 8 = 3

Do you ever think of a negative number?

45. Weighing Meat I

Start from the small numbers.

First we need a 1 lb weight because we have to weigh a 1 lb piece of meat.

Now a 2 lb piece of meat comes. We can make another 1 lb weight to do the job. But we may make a 2 lb weight.

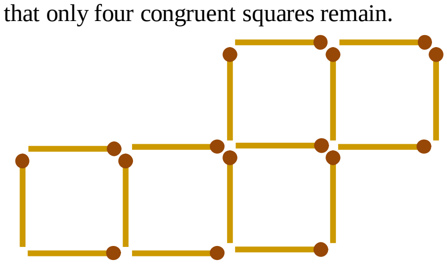
Combining the 1 lb weight and the 2 lb weight we can weigh a 3 lb piece of meat.

Go ahead.

Creative Thinking Problems 46 to 48

46. Five Square to Four

Matchsticks are arranged to form five congruent squares as shown. Move two matchsticks to change the figure such that only four congruent squares remain.



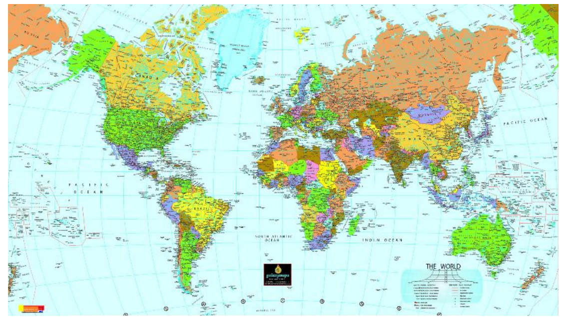
47. All Digits from 1 to 9

Different letters represent different digits. Find digits for *A* to *J* such that both expressions are true.

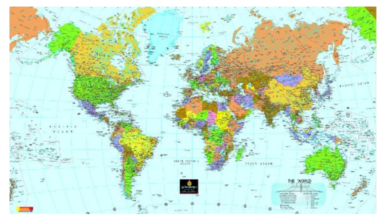
$$\begin{array}{r} ABC \\ +\quad DE \\ \hline FGHI \end{array} \qquad \begin{array}{r} ABC \\ -\quad DE \\ \hline EJD \end{array}$$

48. Two Maps

I have a large map called the *L* map.

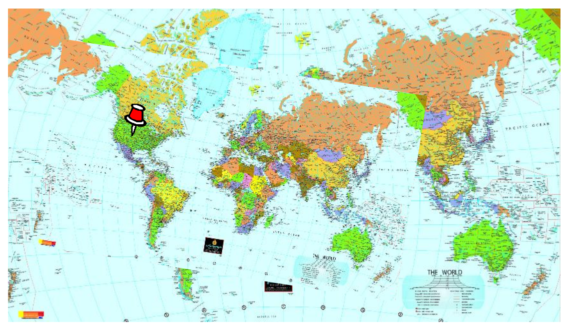


I have a small map called the *S* map.

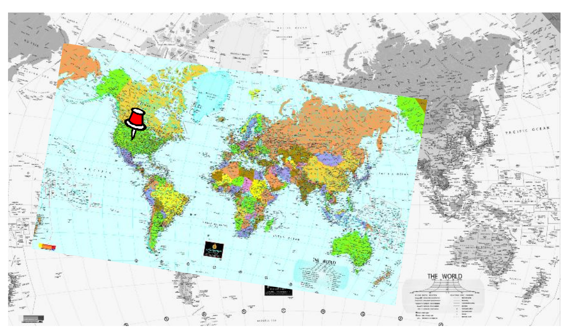


The two maps are the same except their sizes.

I arbitrarily toss the *S* map onto the *L* map such that the *S* map is completely inside the *L* map.



To make it clearer the *L* map is grayed in the figure below.



Prove that there is a point on the *S* map through which you pin such that the pinned point on the *L* map represents the same place as that on the *S* map.

It is not important that you find the place, but you have to prove that it exists.

(Clues and solutions will be given in the next issues.)