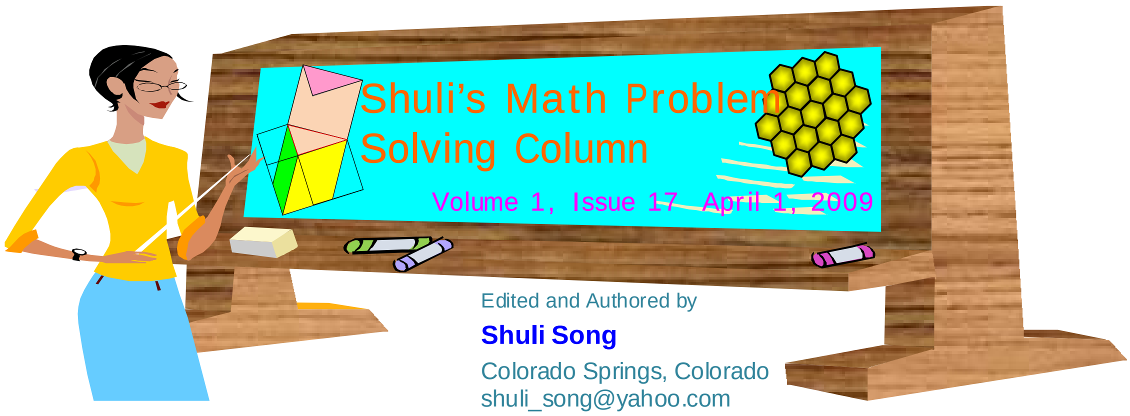


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## Content



## Math Trick

### Mental Calculation: $\overline{99a} \times \overline{99b}$

### The Trick

Mentally calculate:

$997 \times 998 =$ $\quad 996 \times 993 =$ $\quad 995 \times 992 =$

$999 \times 994 =$ $\quad 991 \times 992 =$ $\quad 998 \times 995 =$

To calculate the product of two numbers close to 1000, we have a short cut.

Write the multiplications in the general form: $\overline{99a} \times \overline{99b}$ where $a$ and $b$ are digits.

Let $c = 1000 - \overline{99a}$ and $d = 1000 - \overline{99b}$.

Then $\overline{99a} \times \overline{99b} = (1000 - c) \times (1000 - d)$.

The steps are shown through the following examples.

*Example 1*

Calculate $993 \times 998$.

*Step 1*: Calculate $c = 1000 - \overline{99a}$ and $d = 1000 - \overline{99b}$.

In this example, $c = 1000 - 993 = 7$ and $d = 1000 - 998 = 2$.

*Step 2*: Calculate $\overline{99a} - d$ or $\overline{99b} - c$.

In this example, $\overline{99a} - d = 993 - 2 = 991$ or $\overline{99b} - c = 998 - 7 = 991$.

*Step 3*: Calculate $c \times d$.

In this example, $7 \times 2 = 14$.

*Step 4*: Attach the result in step 3 as three digits to the right of the result in step 2.

In this example, attach 014 to the right of 991: 991014.

Now we are done: $993 \times 998 = 991014$.

*Example 2*

Calculate $998 \times 997$.

*Step 1*: Calculate $1000 - 998 = 2$ and $1000 - 997 = 3$.

*Step 2*: Calculate $998 - 3 = 995$ or $997 - 2 = 995$.

*Step 3*: Calculate $2 \times 3 = 6$, treated as three digits: 006

*Step 4*: Attach 006 to the right of 995: 995006.

We have $997 \times 998 = 995006$.

This works for two numbers not so close to 1000.

*Example 3*

Calculate $989 \times 993$.

*Step 1*: Calculate $1000 - 989 = 11$ and $1000 - 993 = 7$.

*Step 2*: Calculate $989 - 7 = 982$ or $993 - 11 = 982$.

*Step 3*: Calculate $11 \times 7 = 77$, treated as three digits: 077

*Step 4*: Attach 077 to the right of 982: 982077.

We obtain $989 \times 993 = 982077$.

*Example 4*

Calculate $986 \times 989$.

*Step 1*: Calculate $1000 - 986 = 14$ and $1000 - 989 = 11$.

*Step 2*: Calculate $986 - 11 = 975$ or $989 - 14 = 975$.

*Step 3*: Calculate $14 \times 11 = 154$.

*Step 4*: Attach 154 to the right of 975: 975154.

So $986 \times 989 = 975154$.

*Example 5*

Calculate $897 \times 989$.

*Step 1*: Calculate $1000 - 897 = 103$ and $1000 - 989 = 11$.

*Step 2*: Calculate $897 - 11 = 886$ or $989 - 103 = 886$.

*Step 3*: Calculate $103 \times 11 = 1133$.

*Step 4*: Add 1, the thousands digit of 1133, to 886 yielding 887, and attach 133 to the right of 887.

We obtain $897 \times 989 = 887133$.

*Example 6*

Calculate $896 \times 893$.

*Step 1*: Calculate $1000 - 896 = 104$ and $1000 - 893 = 107$.

*Step 2*: Calculate $893 - 104 = 789$ or $896 - 107 = 789$.

*Step 3*: Calculate $104 \times 107 = 11128$.

Recall how to calculate $\overline{10a} \times \overline{10b}$ mentally.

*Step 4*: Add 11, the first two digits of 11128, to 789 yielding 800, and attach 128 to the right of 800.

We have $896 \times 893 = 800128$.

## Why Does This Work?

$$\overline{99a} \times \overline{99b} = (1000 - c) \times (1000 - d)$$
$$= 1000000 - 1000c - 1000d + cd$$
$$= 1000\,(1000 - c - d) + cd = 1000(\overline{99a} - d) + cd$$

or
$$= 1000(\overline{99b} - c) + cd$$

This shows that to calculate $\overline{99a} \times \overline{99b}$, we may do

*Step 1:* Calculate $c = 1000 - \overline{99a}$ and $d = 100 - \overline{99b}$.

*Step 2:* Calculate $\overline{99a} - d$ or $\overline{99b} - c$.

*Step 3:* Calculate $c \times d$.

*Step 4:* Attach $c \times d$ as THREE digits to the right of $\overline{99a} - d$ or $\overline{99b} - c$.

## Practice Problems I

| | | |
|---|---|---|
| $996 \times 997 =$ | $991 \times 998 =$ | $993 \times 995 =$ |
| $991 \times 996 =$ | $995 \times 996 =$ | $999 \times 992 =$ |
| $992 \times 994 =$ | $994 \times 998 =$ | $997 \times 992 =$ |
| $998 \times 993 =$ | $993 \times 999 =$ | $997 \times 995 =$ |
| $997^2 =$ | $994^2 =$ | $998^2 =$ |

## Practice Problems II

| | | |
|---|---|---|
| $986 \times 997 =$ | $991 \times 987 =$ | $993 \times 985 =$ |
| $981 \times 996 =$ | $995 \times 986 =$ | $989 \times 992 =$ |
| $987 \times 984 =$ | $986 \times 988 =$ | $987 \times 982 =$ |
| $988 \times 983 =$ | $989 \times 985 =$ | $984 \times 983 =$ |
| $987^2 =$ | $986^2 =$ | $988^2 =$ |

## Practice Problems III

| | | |
|---|---|---|
| $896 \times 997 =$ | $891 \times 989 =$ | $893 \times 995 =$ |
| $892 \times 891 =$ | $895 \times 898 =$ | $893 \times 899 =$ |
| $895 \times 897 =$ | $889 \times 899 =$ | $894 \times 893 =$ |
| $887 \times 899 =$ | $885 \times 884 =$ | $890 \times 873 =$ |

# Math Competition Skill

## Systematically Listing According to Numbers

In math competitions at elementary and junior high levels, the most counting problems can be solved by systematically listing. In the previous issues we have used this method in many examples.

We may systematically list objects according to numbers or shapes. Sometimes we should list according to both of them.

This short lesson will demonstrate systematically listing according to numbers.

When we list objects, we should list from the smallest to the greatest, or from the greatest to the smallest.

## Examples

*Example 1*

How many three-digit numbers can we make with digits 1, 2, and 3 if

(a) no digit can be repeated;

(b) digits may be repeated?

*Answer:* (a) 6; (b) 27

*Solution:*

(a) We can make 6 numbers: 123, 132, 213, 231, 312, 321.

(b) With 1 as the leftmost digit we have 9 numbers: 111, 112, 113, 121, 122, 123, 131, 132, 133.

With 2 as the leftmost digit we have 9 numbers: 211, 212, 213, 221, 222, 223, 231, 232, 233.

With 3 as the leftmost digit we have 9 numbers: 311, 312, 313, 321, 322, 323, 331, 332, 333.

Altogether, we can make 27 numbers.

*Example 2*

How many ways are there to express 10 as a sum of 3 positive integers (not necessarily distinct). Two expressions will be considered the same if they include the same numbers in different orders.

*Answer:* 8

*Solution:*

We list as follows:

The smallest number is 1:

$1+1+8$ $1+2+7$ $1+3+6$ $1+4+5$

The smallest number is 2:

$2+2+6$ $2+3+5$ $2+4+4$

The smallest number is 3:

$3+3+4$

There are 8 ways altogether.

*Example 3*

Roll two standard dices. What is the probability that the sum of two numbers is divisible by 3?

*Answer:* $\frac{1}{3}$

*Solution:*

The sum divisible by 3 can be 3, 6, 9, or 12. We list in the table:

| Dice 1 | Dice 2 | Sum |
|---|---|---|
| 1 | 2 | 3 |
| 2 | 1 | 3 |
| 1 | 5 | 6 |
| 2 | 4 | 6 |
| 3 | 3 | 6 |
| 4 | 2 | 6 |
| 5 | 1 | 6 |
| 3 | 6 | 9 |
| 4 | 5 | 9 |
| 5 | 4 | 9 |
| 6 | 3 | 9 |
| 6 | 6 | 12 |

There are $6\times 6=36$ possibilities in rolling two dices, and 12 possibilities that the sum is divisible by 3. The probability is $\frac{12}{36}=\frac{1}{3}$.

*Example 4*

How many ways are there to have 30 cents with US coins?

*Answer:* 18

*Solution:*

We list in the table:

| Quarter | Dime | Nickle | Penny |
|---|---|---|---|
| 1 | | 1 | |
| 1 | | | 5 |
| | 3 | | |
| | 2 | 2 | |
| | 2 | 1 | 5 |
| | 2 | | 10 |
| | 1 | 4 | |
| | 1 | 3 | 5 |
| | 1 | 2 | 10 |
| | 1 | 1 | 15 |
| | 1 | | 20 |
| | | 6 | |
| | | 5 | 5 |
| | | 4 | 10 |
| | | 3 | 15 |
| | | 2 | 20 |
| | | 1 | 25 |
| | | | 30 |

There are 18 combinations to have 30 cents.

*Example 5*

All sides of an isosceles triangle are integral. The perimeter is 2008. How many triangles satisfy the conditions?

*Answer:* 501

*Solution:*

Let $a$, which is an integer, be the length of two equal sides. Then the length of the base is

$$2008-2a=2\cdot(1004-a).$$

So the length of the base must be an even positive integer. The even positive integers start from 2. So the possible smallest length of the base is 2. The length of the base must be less than $\frac{2008}{2}=1004$. So the possible greatest length of the base is 1002. Then the length of the base could be 2, 4, up to 1002.

Therefore, there are $\frac{1002-2}{2}+1=501$ triangles.

*Example 6*

The three sides of a triangle are all integral. One side and only one side is 5, and it is not the shortest side. How many triangles are there satisfying the conditions?

*Answer:* 10

*Solution:*

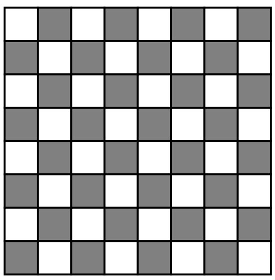
*Case 1*: The longest side is 5. The other two sides may be 2, 4; 3, 3; 3, 4; and 4, 4, listed from the smallest to the greatest for one side. There are four triangles of this kind.

*Case 2*: The medium side is 5. The other two sides may be 2, 6; 3, 6; 3, 7; 4, 6; 4, 7; and 4, 8. There are six triangles of this kind.

Altogether there are $4+6=10$ triangles.

*Example 7*

How many squares of all sizes can you count in a standard chessboard?



*Answer:* 204

*Solution:*

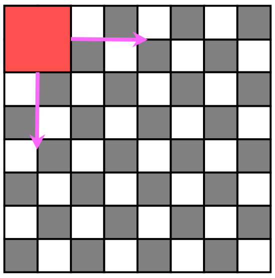
The smallest square is $1\times1$, and the greatest square is $8\times8$.

Obviously the number of $1\times1$ squares is $8\times8=64$.

For the $2\times2$ squares, we may obtain the answer by "*Moving the Shape*".

Place a $2\times2$ square at the left-top corner. Then there are 7 positions including the current position to move it to the right, and 7 positions to move it to the bottom. So the number of $2\times2$ squares is $7\times7=49$.

Similarly, the number of $3\times3$ squares is $6\times6=36$.



We have the following table:

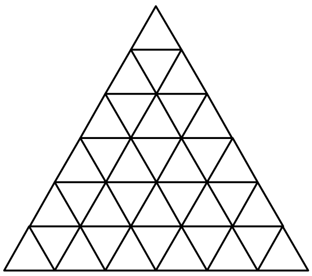
| Kind of Squares | Number of Squares |
|---|---|
| $1\times1$ | $8\times8=64$ |
| $2\times2$ | $7\times7=49$ |
| $3\times3$ | $6\times6=36$ |
| $4\times4$ | $5\times5=25$ |
| $5\times5$ | $4\times4=16$ |
| $6\times6$ | $3\times3=9$ |
| $7\times7$ | $2\times2=4$ |
| $8\times8$ | $1\times1=1$ |

The total number of squares of all sizes is

$$1+4+9+16+25+36+49+64=204.$$

*Example 8*

How many triangles of all sizes are there in the figure below.



*Answer:* 78

*Solution:*

We call the smallest triangles the *unit triangles*. We list according to the number of unit triangles contained in a triangle.

*Type 1:* Containing one unit triangle.

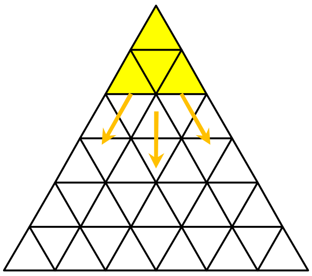
There are 36 triangles of this type.

*Type 2:* Containing 4 unit triangles.

There are two different shapes: a. triangles pointing up; b. triangles pointing down.

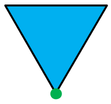
For the triangles pointing up, we use "*Moving the Shape*" to count.

Place a triangle at the top of the figure.

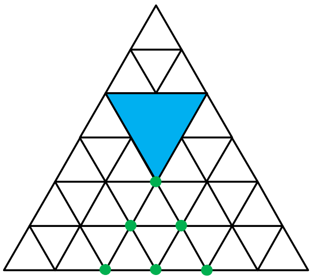


There are $1+2+3+4+5=15$ different positions to move it.

For the triangles pointing down, we apply "*Using the Reference*" to count. We use the vertex pointing the bottom as the reference.



There are $1+2+3=6$ points each of which can be the reference.



Altogether there are $15+6=21$ triangles containing 4 unit triangles.

*Type 3:* Containing 9 unit triangles.

There are $1+2+3+4=10$ triangles pointing up, and one triangle pointing down.

Altogether there are $10+1=11$ triangles containing 9 unit triangles.

*Type 4:* Containing 16 unit triangles.

There are $1+2+3=6$ triangles pointing up, and no triangle pointing down.

*Type 5:* Containing 25 unit triangles.

There are $1+2=3$ triangles pointing up, and no triangle pointing down.

*Type 6:* Containing 36 unit triangles.

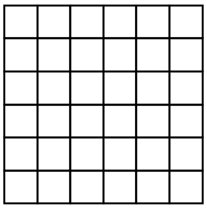
There is only one triangle pointing up.

The total number of triangles of all sizes is

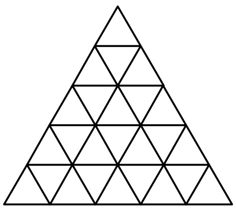
$$36+21+11+6+3+1=78.$$

## Practice Problems I

1. How many four-digit numbers can we make with digits 1, 2, 3, and 4 if no digit may be repeated?
2. How many ways are there to express 10 as a sum of 4 positive integers (not necessarily distinct). Two expressions will be considered the same if they include the same numbers in different orders.
3. Roll two standard dices. What is the probability that the sum of two numbers is a prime?
4. How many ways are there to have 35 cents with US coins?
5. All sides of an isosceles triangle are integral. The perimeter is 1001. How many triangles are there satisfying the conditions?
6. The three sides of a triangle are all integral. One side and only one side is 6, and it is not the shortest side. How many triangles satisfy the conditions?
7. How many squares of all sizes can you count in the $6\times6$ grid?

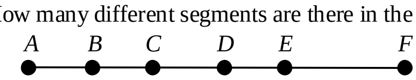


8. How many triangles of all sizes are there in the figure.

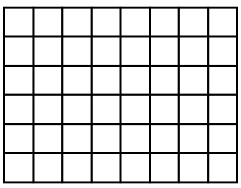


## Practice Problems II

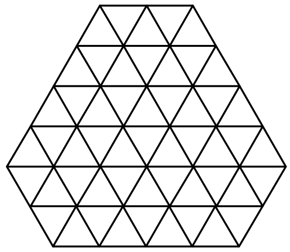
1. How many different segments are there in the figure?




2. How many squares of all sizes can you count in the $6\times8$ grid?



3. The three sides of a triangle are all integral. The perimeter is 18. How many triangles are there satisfying the conditions?
4. The three sides of a scalene triangle are all integral. The perimeter is 21. How many triangles are there satisfying the conditions?
5. How many triangles of all sizes are there is the figure.



## A Problem from a Real Math Competition

Today's problem comes from MathCounts.

(MathCounts 1992 National Sprint, Problem 24)

What is the greatest number of bags that can be used to hold 190 marbles if each bag must contain at least one marble, but no two bags may contain the same number of marbles?

*Answer:* 19

*Solution:*

Since we want to have the greatest number of bags, we place as few marbles as possible in the bags. We put 1 marble in the first bag, 2 marbles in the second, etc.

Since $1+2+\cdots+19=190$, we can have 19 bags at most.

## Practice Problems

A monkey puts 1000 peanuts onto dishes. What is the greatest number of dishes the monkey can use to hold 1000 peanuts if no dish is empty, and no two dishes contain the same number of peanuts?

## Answers to All Practice Problems in Last Issue

### Math Trick: Mental Calculation

### Practice Problems I

| | | |
|---|---|---|
| 9312 | 8918 | 8835 |
| 8736 | 9120 | 9108 |
| 8648 | 9121 | 8924 |
| 9114 | 9207 | 9215 |
| 9409 | 9216 | 8836 |

### Practice Problems II

| | | |
|---|---|---|
| 8342 | 7917 | 7905 |
| 7736 | 8170 | 8188 |
| 7308 | 7568 | 7134 |
| 7304 | 7565 | 6972 |
| 7569 | 7396 | 7056 |

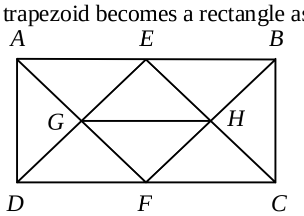
### How Do You Count? –Rectangles in Tableaus

$f(7)=210$

## A Problem from a Real Math Competition

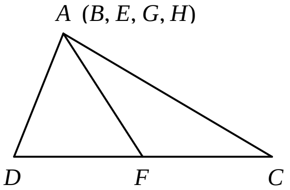
If $a = b$, the trapezoid becomes a rectangle as shown.




Then $GH = \frac{a}{2}$ or $\frac{b}{2}$.

Only choices A and E possess this property.

If $a = 0$, the trapezoid becomes a triangle as shown.




*B*, *E*, *G*, and *H* are all overlapped with *A*. So $GH = 0$. Only E has this property in the remaining two choices.

Therefore, the answer is E.

## Solutions to Creative Thinking Problems 43 to 45

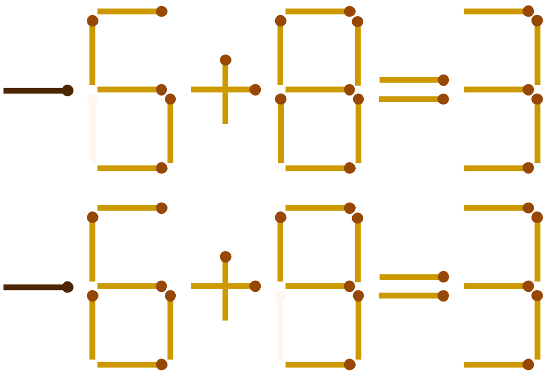
### 43. Liar, Truth-teller, and Neutral

Assume that the inhabitant is a liar. When he says "I'm not a truth-teller", he is telling the truth. It is a contradiction. Assume that the inhabitant is a truth-teller. When he says "I'm not a truth-teller", he is lying. It is a contradiction too.

Therefore, the inhabitant is a neutral.

### 44. 6 + 8 = 3

These are my two solutions:



### 45. Weighing Meat I

Continue on the clue in the last issue. We have two weights of 1 lb. and 2 lb respectively.

For a piece of meat of 4 lb., we make a new weight of 4 lb. Then we can weigh up to 7 lb. by combining the existing weights.

Now we need a weight of 8 lb. for a piece of meat of 8 lb.

We obtain the pattern: the next weight is double the previous one.

So seven weights are enough: 1, 2, 4, 8, 16, 32, and 64. We can actually weigh up to 127 lb with these weights.

## Clues to Creative Thinking Problems 46 to 48

### 46. Five Square to Four

Count the number of matchsticks.

### 47. All Digits from 1 to 9

You don't need a clue in this problem, do you?

### 48. Two Maps

The true proof may be far away from the scope of the column. Find a way to convince people or yourself to believe that it is true.

## Creative Thinking Problems 49 to 51

### 49. 9 – 1 = 10

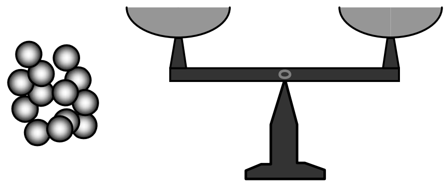
How do you take 1 away from 9 but leave 10?

### 50. Who Is Taller?

One hundred students are positioned in a 10×10 matrix. From each of the 10 columns thus formed the shortest student is selected, and the tallest of these 10 shortest students is tagged *TS*. These students now return to their initial positions. Then the tallest student in each row is selected, and among these 10 tallest students the shortest is tagged *ST*. If no two students have the same height, who is taller – *TS* or *ST*?

### 51. 12 Balls

There are 12 balls that look exactly the same. One out of the 12 balls is bad. All good balls have the same weight, but the bad ball has a slightly different weight. Identify the bad ball and point out whether the bad ball is lighter or heavier by three times of weighing with a pan scale.



(Clues and solutions will be given in the next issues.)