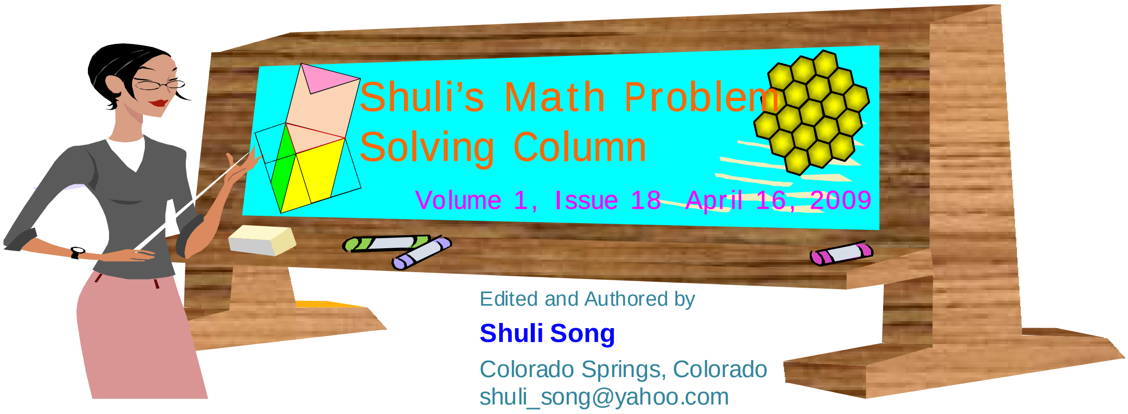


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## Content



## Math Trick

### Mental Calculation: $\overline{19a} \times \overline{19b}$

#### The Trick

Mentally calculate:

$197 \times 198 =$   $196 \times 193 =$   $195 \times 192 =$

$199 \times 194 =$   $191 \times 192 =$   $198 \times 195 =$

In each multiplication the two numbers are close to 200.

Write the multiplications in the general form: $\overline{19a} \times \overline{19b}$ where $a$ and $b$ are digits.

Let $c = 200 - \overline{19a}$ and $d = 200 - \overline{19b}$. Then

$$\overline{19a} \times \overline{19b} = (200-c) \times (200-d).$$

The short cut is shown through the following examples.

*Example 1*

Calculate $193 \times 198$.

*Step 1*: Calculate $c = 200 - \overline{19a}$ and $d = 200 - \overline{19b}$.

In this example, $c = 200 - 193 = 7$ and $d = 200 - 198 = 2$.

*Step 2*: Calculate $\overline{19a} - d$ or $\overline{19b} - c$.

In this example, $\overline{19a} - d = 193 - 2 = 191$ or $\overline{19b} - c = 198 - 7 = 191$.

*Step 3*: Calculate $2 \times (\overline{19a} - d)$ or $2 \times (\overline{19b} - c)$.

In this example, $2 \times 191 = 382$.

*Step 4*: Calculate $c \times d$.

In this example, $7 \times 2 = 14$.

*Step 5*: Attach the result in step 4 as two digits to the right of the result in step 3.

In this example, attach 14 to the right of 382: 38214.

Now we are done: $193 \times 198 = 38214$.

*Example 2*

Calculate $198 \times 197$.

*Step 1*: Calculate $200 - 198 = 2$ and $200 - 197 = 3$.

*Step 2*: Calculate $198 - 3 = 195$ or $197 - 2 = 195$.

*Step 3*: Calculate $195 \times 2 = 390$.

*Step 4*: Calculate $2 \times 3 = 6$, treated as two digits: 06

*Step 5*: Attach 06 to the right of 390: 39006.

We have $198 \times 197 = 39006$.

This works for two numbers not so close to 200.

*Example 3*

Calculate $189 \times 193$.

*Step 1*: Calculate $200 - 189 = 11$ and $200 - 193 = 7$.

*Step 2*: Calculate $189 - 7 = 182$ or $193 - 11 = 182$.

*Step 3*: Calculate $182 \times 2 = 364$.

*Step 4*: Calculate $11 \times 7 = 77$.

*Step 5* Attach 77 to the right of 364: 36477.

We obtain $189 \times 193 = 36477$.

*Example 4*

Calculate $186 \times 189$.

*Step 1*: Calculate $200 - 186 = 14$ and $200 - 189 = 11$.

*Step 2*: Calculate $186 - 11 = 175$ or $189 - 14 = 175$.

*Step 3*: Calculate $175\times 2=350$.

*Step 4*: Calculate $14\times 11=154$.

*Step 5*: Add 1 to 350 yielding 351, and attach 54 to the right of 351: 35154.

We have $186\times 189=35154$.

## Why Does This Work?

$$\overline{19a}\times\overline{19b}=(200-c)\times(200-d)=40000-200c-200d+cd$$
$$=200(200-c-d)+cd=200(\overline{19a}-d)+cd$$
or
$$=200(\overline{19b}-c)+cd$$

This shows that to calculate $\overline{19a}\times\overline{19b}$, we may do

*Step 1:* Calculate $c=200-\overline{19a}$ and $d=200-\overline{19b}$.

*Step 2:* Calculate $\overline{19a}-d$ or $\overline{19b}-c$.

*Step 3*: Calculate $2\times(\overline{19a}-d)$ or $2\times(\overline{19b}-c)$.

*Step 4:* Calculate $c\times d$.

*Step 5:* Attach $c\times d$ as TWO digits to the right of $2\times(\overline{19a}-d)$ or $2\times(\overline{19b}-c)$.

## Mental Calculation: $\overline{n9a}\times\overline{n9b}$

The similar procedure applies to the multiplications in the form $\overline{n9a}\times\overline{n9b}$ where $n$ is a digit greater than 2. Instead of 2 we have to multiply $\overline{n9a}-d$ or $\overline{n9b}-c$ by $n+1$ in step 3.

*Example 5*

Calculate $296\times 297$.

*Step 1*: Calculate $300-296=4$ and $300-297=3$.

*Step 2*: Calculate $296-3=293$ or $297-4=293$.

*Step 3*: Calculate $293\times 3=879$.

*Step 4*: Calculate $4\times 3=12$

*Step 5*: Attach 12 to the right of 879: 87912.

Then $296\times 297=87912$.

*Example 6*

Calculate $388\times 393$.

*Step 1*: Calculate $400-388=12$ and $400-393=7$.

*Step 2*: Calculate $388-7=381$ or $393-12=381$.

*Step 3*: Calculate $381\times 4=1524$.

*Step 4*: Calculate $12\times 7=84$.

*Step 5* Attach 84 to the right of 1524: 152484.

We have $388\times 393=152484$.

*Example 7*

Calculate $586\times 588$.

*Step 1*: Calculate $600-586=14$ and $600-588=12$.

*Step 2*: Calculate $586-12=574$ or $588-14=574$.

*Step 3*: Calculate $574\times 6=3444$.

*Step 4*: Calculate $14\times 12=168$.

*Step 5*: Add 1 to 3444 yielding 3445, and attach 68 to the right of 3445: 344568.

So $586\times 588=344568$.

## Practice Problems I

| | | |
|---|---|---|
| $196\times 197=$ | $191\times 198=$ | $193\times 195=$ |
| $191\times 196=$ | $195\times 196=$ | $199\times 192=$ |
| $192\times 194=$ | $194\times 198=$ | $197\times 192=$ |
| $198\times 193=$ | $193\times 199=$ | $197\times 195=$ |
| $197^2=$ | $196^2=$ | $194^2=$ |

## Practice Problems II

| | | |
|---|---|---|
| $186\times 197=$ | $191\times 187=$ | $193\times 185=$ |
| $181\times 196=$ | $195\times 186=$ | $189\times 192=$ |
| $187\times 184=$ | $186\times 188=$ | $187\times 182=$ |
| $188\times 183=$ | $189\times 185=$ | $184\times 183=$ |
| $188^2=$ | $187^2=$ | $183^2=$ |

## Practice Problems III

| | | |
|---|---|---|
| $296\times 297=$ | $287\times 297=$ | $289\times 288=$ |
| $391\times 396=$ | $385\times 396=$ | $387\times 386=$ |
| $492\times 494=$ | $498\times 487=$ | $486\times 488=$ |
| $598\times 593=$ | $599\times 583=$ | $697^2=$ |
| $689\times 693=$ | $796^2=$ | $898\times 895=$ |

# Math Competition Skill

## Systematically Listing According to Shapes

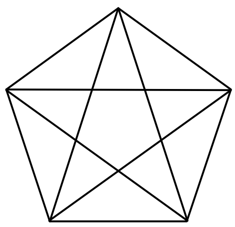
We have practiced systematically listing according to shapes in counting parallelograms in triangular grids (*Issue 10, Volume 1*) and counting rectangles in tableaus (*Issue 16, Volume 1*). This short lesson will present more examples.

## Examples

*Example 1*

(MathCounts State Sprint 1993 Problem 6)

How many triangles of all sizes can you count in the figure below?



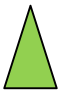
 

*Answer:* 35

*Solution:*

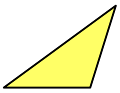
We can classify the triangles into six types.

*Type One:*



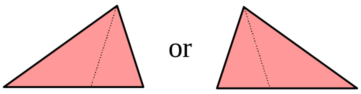
There are 5 triangles.

*Type Two:*



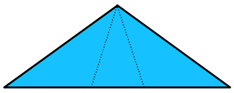
There are 5 triangles.

*Type Three:*



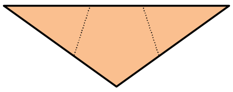

There are 10 triangles.

*Type Four:*



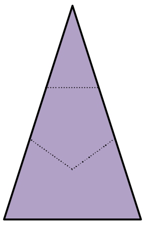
There are 5 triangles.

*Type Five:*



There are 5 triangles.

*Type Six:*



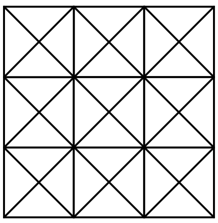
There are 5 triangles.

Altogether, the number of triangles is

$$5+5+10+5+5+5=35.$$

*Example 2*

How many squares of all sizes can you count in the figure?

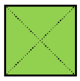


*Answer:* 31

*Solution:*

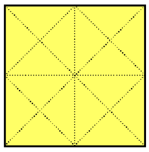
We can categorize the squares into five types.

*Type One:*



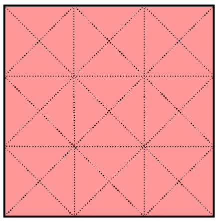
There are 9 squares.

*Type Two:*



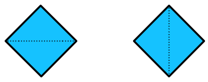
There are 4 squares.

*Type Three:*



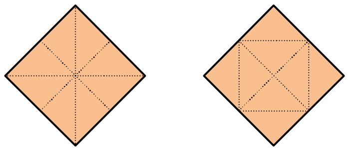
There is only one square.

*Type Four:*



There are 12 squares.

*Type Five:*



There are 5 squares.

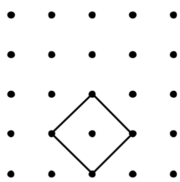
Altogether, the number of squares is

$$9+4+1+12+5=31.$$

*Example 3*

(2003-2004 MathCounts Handbook Work-Out 8 Problem 8)

On this 5 by 5 grid of dots, one square is shown in the diagram. Including this square, how many squares of different sizes can be counted using four dots of this array as vertices?



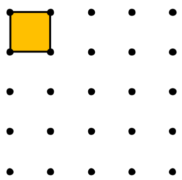
*Answer:* 50

*Solution:*

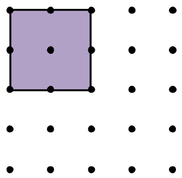
 

There are 8 types of squares. For each type we can obtain the number of squares using the "*Moving the Shape*" method.

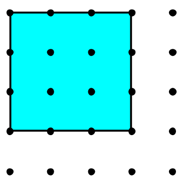
Obviously there are $4\times 4=16$ squares of $1\times 1$.



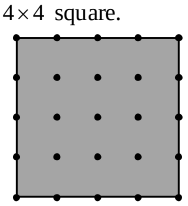
We place a $2\times 2$ square at the left-top corner. There are three positions including the current position to move it to the right and three positions to move it to the bottom. So the number of $2\times 2$ squares is $3\times 3=9$.



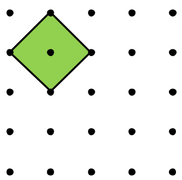
Similarly we can obtain the number of $3\times 3$ squares, which is $2\times 2=4$.



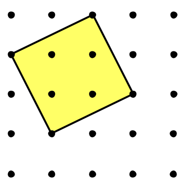
There is only one $4\times 4$ square.

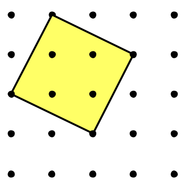


We place a $\sqrt{2}\times\sqrt{2}$ square at the left-top corner. Using the "*Moving the Shape*" method we know that the number of squares in this type is $3\times 3=9$.



There are two orientations of $\sqrt{5}\times\sqrt{5}$ squares, which are shown below.

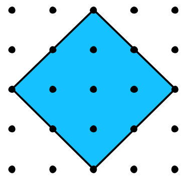




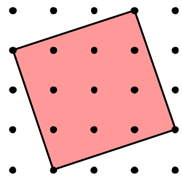
For each orientationthere are $2\times 2=4$ squares.

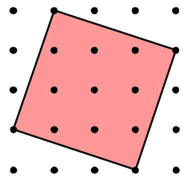
Altogether, there are $2\times 4=8$ squares of this type.

The following figure shows the only one $\sqrt{8}\times\sqrt{8}$ square.



There are also two orientations of $\sqrt{10}\times\sqrt{10}$ squares, which are shown below.





There is only one in each orientation.

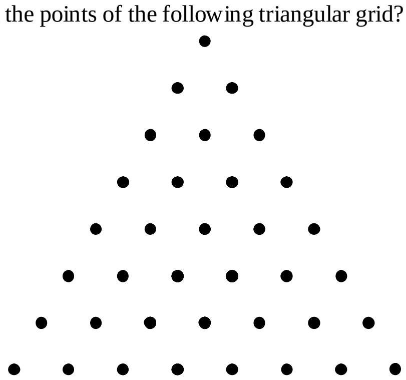
So there are two $\sqrt{10}\times\sqrt{10}$ squares.

Therefore, the total number of squares is

$$16+9+4+1+9+8+1+2=50.$$

*Example 4*

How many regular hexagons are there whose vertices are among the points of the following triangular grid?

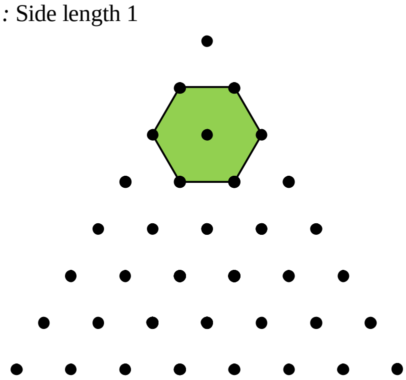


*Answer:* 21

*Solution:*

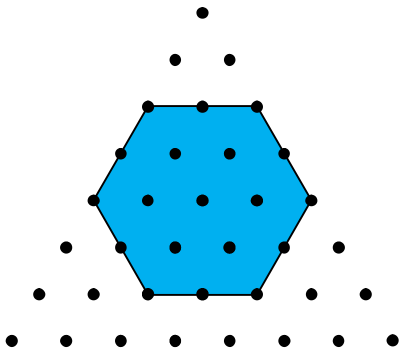
There are three types of regular hexagons.

*Type 1:* Side length 1



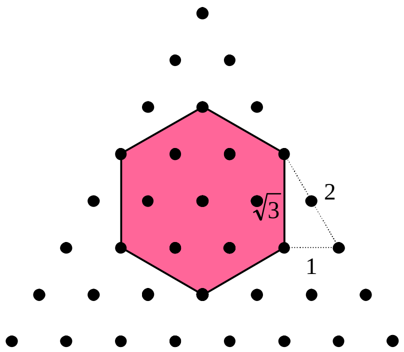
With the "*Moving the Shape*" method, we can find 15 regular hexagons of this type.

*Type 2:* Side length 2



We can count 3 regular hexagons of this type by "*Moving the Shape*".

*Type 3:* Side length $\sqrt{3}$

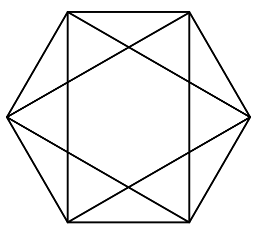



We can also find 3 regular hexagons of this type with the same way.

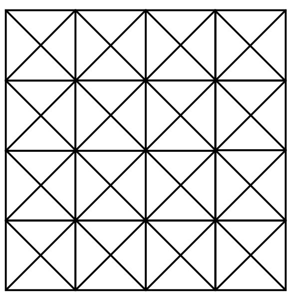
Altogether, there are $15+3+3=21$ regular hexagons.

## Practice Problems I

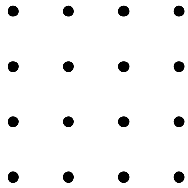
1. How many triangles of all sizes can you count in the figure?



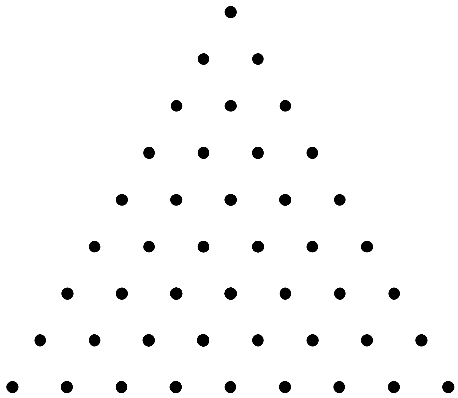
2. How many squares of all sizes can you count in the figure?



3. Sixteen points are equally spaced as shown. How many sets of four points are the vertices of a square?



4. How many regular hexagons are there whose vertices are among the points of the following triangular grid?

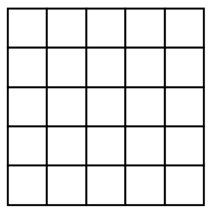


## Practice Problems II

(All problems are from MathCounts)

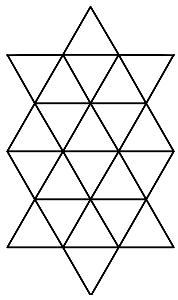
1. (1990 MathCounts National Team Problem 1)

How many squares are contained in the figure?



2. (1986 MathCounts State Individual Problem 2)

How many triangles of any size are contained in the figure shown?

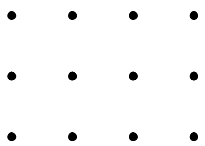


3. (2001 MathCounts State Team Problem 10)

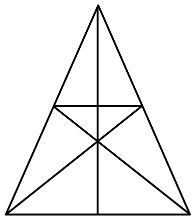
How many equilateral triangles can be formed within the same plane using at least two vertices that are also vertices of a given regular hexagon?

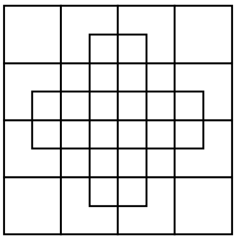
4. (2000 MathCounts Chapter Sprint Problem 14)
How many different squares can be formed by using four of the evenly-spaced dots blow as vertices of the square?



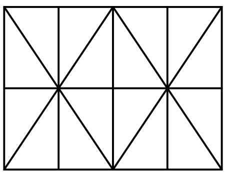
5. (2000 MathCounts State Sprint Problem 13)
How many triangles are in the figure?



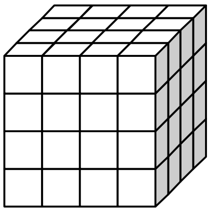
6. (1999 MathCounts National Sprint Problem 22)
How many squares are pictured?



7. (2005 MathCounts Chapter Team Problem 4)
How many triangles are in the figure?



8. (2001 MathCounts State Sprint Problem 8)
Sixty-four unit cubes are placed together to create a large cube. How many cubes with integer dimensions are in the $4\times4\times4$ cube?



## A Problem from a Real Math Competition

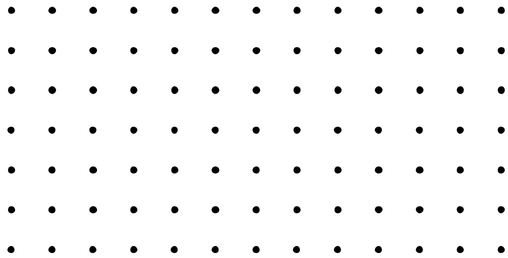
Today's problem comes from University of Northern Colorado Mathematics Contest (UNCMC).

(UNCMC 2000-2001 Final Round Problem 7)

Two points are randomly and simultaneously selected from the 7 by 13 grid of lattice points

$$\{(m,n): 1\le m\le 13 \text{ and } 1\le n\le 7\}.$$

Determine the probability that the distance between the two points is an integer.



*Answer:* $\frac{193}{819}$

*Solution:*

Systematically listing works well in this problem.

Note that $13\times7=91$.

There are $\binom{91}{2}=4095$ ways to choose two points.

If we choose two points in a vertical line, these two points have an integral distance.

There are $13\times\binom{7}{2}=13\times21=273$ ways to choose two points in vertical lines.

If we choose two points in a horizontal line, these two points have an integral distance.

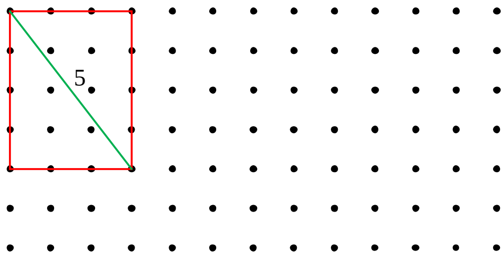
There are $7\times\binom{13}{2}=7\times78=546$ ways to choose two points in horizontal lines.

If two points as the opposite vertices make a $3\times4$ rectangle, the distance between these two points will be 5.

Now we have to count the number of $3\times4$ rectangles whose vertices are in these points.

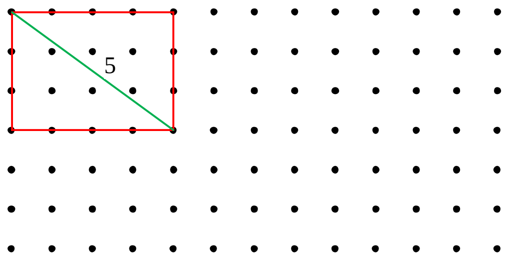
Remember the "*Moving the Shape*" method.

Place a 3 by 4 rectangle at the left-top corner. There are 10 positions including the current position to move it to the right and 3 positions to move it to the bottom. So there are $3\times10=30$ rectangles of $3\times4$, whose vertices are in these points.



We also need to count the number of $4\times3$ rectangles whose vertices are in the lattice points.




Similarly there are $9\times4=36$ rectangles of $4\times3$ whose vertices are in these points.

If two points as the opposite vertices make a $6\times8$ rectangle, the distance between the two points will be 10.

We can count 5 rectangles of $6\times8$.

At last we count the number of $5\times12$ rectangles, the length of whose diagonals is 13.

There are 2 rectangles of $5\times12$.

Altogether we have $30+36+5+2=73$ rectangles with integral diagonals.

Every rectangle has two diagonals. There are $73\times2=146$ diagonals, which are integral.

Therefore, we have $273+546+146=965$ ways to choose two points of an integral distance.

The probability is $\frac{965}{4095}=\frac{193}{819}$.

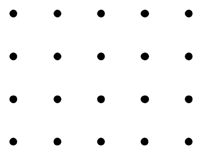
### Practice Problems II

(UNCMC 2000-2001 First Round Problem 10)

Two points are randomly and simultaneously selected from the 4 by 5 grid of 20 lattice points

$$\{(m, n): 1\le m\le5 \text{ and } 1\le n\le4\}.$$

Determine the probability that the distance between the two points is an integer.



## Answers to All Practice Problems in Last Issue

### Math Trick: Mental Calculation

### Practice Problems I

| | | |
|---|---|---|
| 990312 | 989018 | 988035 |
| 987036 | 991020 | 991008 |
| 986048 | 992012 | 989024 |
| 991014 | 992007 | 992015 |
| 994009 | 988036 | 996004 |

### Practice Problems II

| | | |
|---|---|---|
| 983042 | 928117 | 978015 |
| 977076 | 981070 | 981088 |
| 971208 | 974168 | 969234 |
| 971204 | 974165 | 967272 |
| 974169 | 972196 | 976144 |

### Practice Problems III

| | | |
|---|---|---|
| 893312 | 881199 | 888535 |
| 794772 | 803710 | 802807 |
| 802815 | 799211 | 798342 |
| 797413 | 782340 | 776970 |

### Systematically Listing According to Numbers

### Practice Problems I

| | | |
|---|---|---|
| 1. 24 | 2. 9 | 3. 5/12 |
| 3. 24 | 5. 250 | 6. 16 |
| 7. 91 | 8. 48 | |

### Practice Problems II

| | | |
|---|---|---|
| 1. 15 | 2. 133 | 3. 7 |
| 3. 7 | 5. 110 | |

### A Problem from a Real Math Competition

Note that

$$1+2+\cdots+44=990 \text{ and } 1+2+\cdots+45=1035.$$

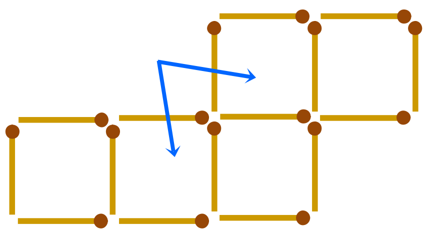
The answer is 44.

The numbers of peanuts on the 44 dishes may be 1, 2, $\cdots$, 42, 43, and 54 respectively.

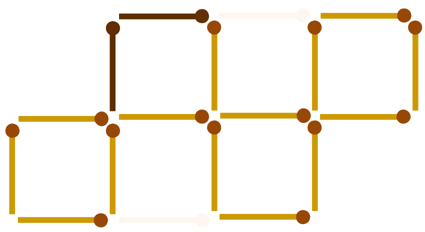
## Solutions to Creative Thinking Problems 46 to 48

### 46. Five Square to Four

We have to make 4 squares with 16 matchsticks in the new shape. We cannot overlap any sides of any squares. So we have to destroy the squares in the old shape, which have the most sides overlapped with others.



Here is the solution:



## 47. Digits from 1 to 9

From the addition, we have $F=1$, $A=9$, and $G=0$.

From the subtraction, we see $E=8$. Now we have

$$\begin{array}{r} 9BC \\ +\quad D8 \\ \hline 10HI \end{array} \qquad \begin{array}{r} 9BC \\ -\quad D8 \\ \hline 8JD \end{array}$$

Note that $I=C+8 \pmod{10}$ and $D=C-8 \pmod{10}$.

So $I-D=6 \pmod{10}$. Thus $I-D=6$ or $D-I=4$

Since 1, 0, 8, and 9 are not available, $I-D=6$ is impossible.

Then $D-I=4$. So $D=6$, $I=2$ or $D=7$, $I=3$.

Case 1: $D=6$ and $I=2$.

Then $C=4$. Now we have

$$\begin{array}{r} 9B4 \\ +\quad 68 \\ \hline 10H2 \end{array} \qquad \begin{array}{r} 9B4 \\ -\quad 68 \\ \hline 8J6 \end{array}$$

In the addition, $B+6+1=10+H$. That is, $B=3+H$. Since only 3, 5, and 7 are available, it is impossible.

Case 2: $D=7$ and $I=3$.

Then $C=5$. Now we have

$$\begin{array}{r} 9B5 \\ +\quad 78 \\ \hline 10H3 \end{array} \qquad \begin{array}{r} 9B5 \\ -\quad 78 \\ \hline 8J7 \end{array}$$

In the addition, $B+7+1=10+H$. That is, $B=2+H$. So $B=6$, $H=4$ or $B=4$, $H=2$.

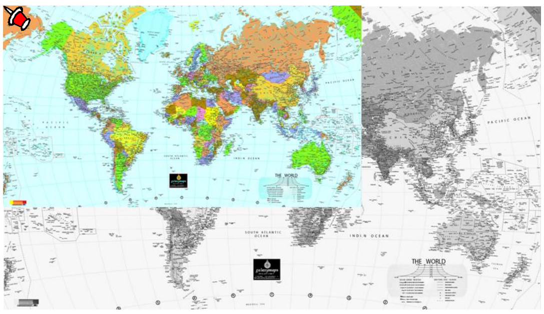
If $B=6$ and $H=4$, then $J=2$. The subtraction cannot be satisfied.

If $B=4$ and $H=2$, then $J=6$. We have the solution:

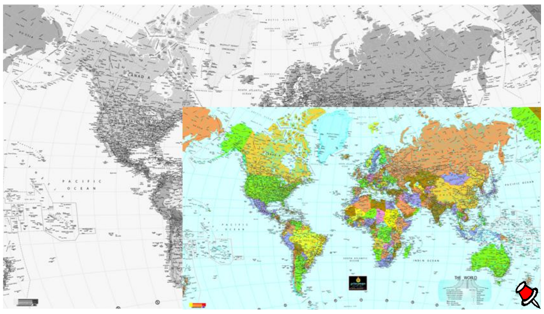
$$\begin{array}{r} 945 \\ +\quad 78 \\ \hline 1023 \end{array} \qquad \begin{array}{r} 945 \\ -\quad 78 \\ \hline 867 \end{array}$$

## 48. Two Maps

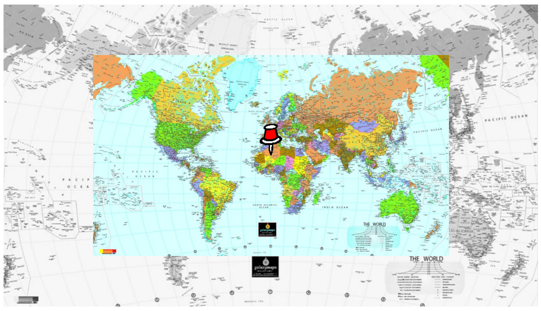
If the two maps are overlapped as shown below, the pin point is the left-top corner.



If the two maps are overlapped as shown in the following figure, the pin point is the right-bottom corner.

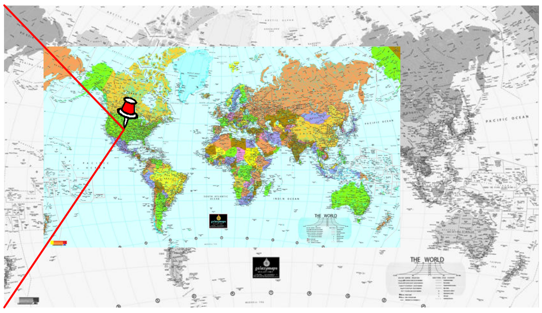


If the *S* map is at the center of the *L* map, the pin point is the center of the two maps.

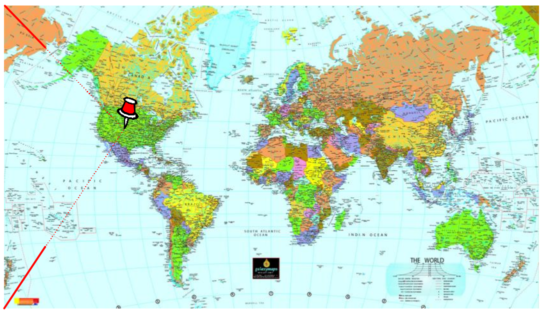


If the sides of the two maps are parallel respectively, we can easily find the pin point.

In the following figure the two maps have the same orientation. Draw the first line through the left-bottom corners of the two maps. Draw the second line through the left-top corners of the two maps. Two lines intersect at a point, at which the same place is represented in both maps. This can readily be proved by similar triangles.

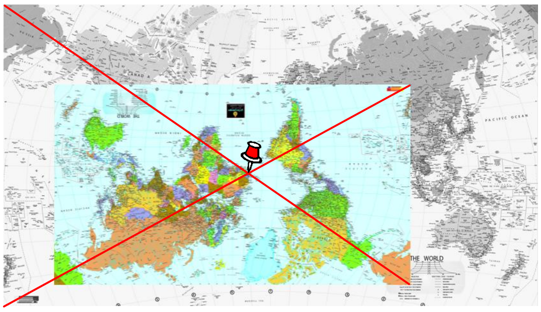


Now I remove the *S* map so that we can recognize the same place in the *L* map.

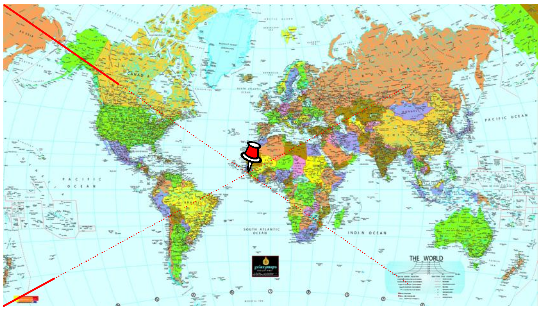


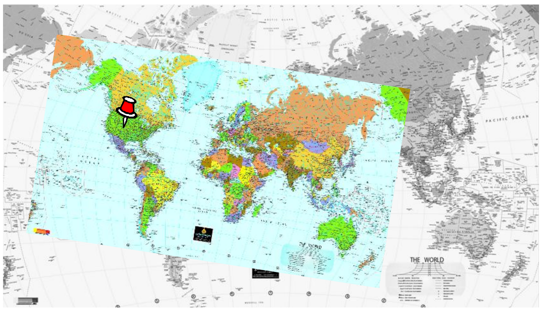
In the figure below the sides of the two maps are parallel, but the $S$ map is upside-down. We can also find the pin point as shown.



Again I remove the $S$ map. Now we can see the same place in the $L$ map.

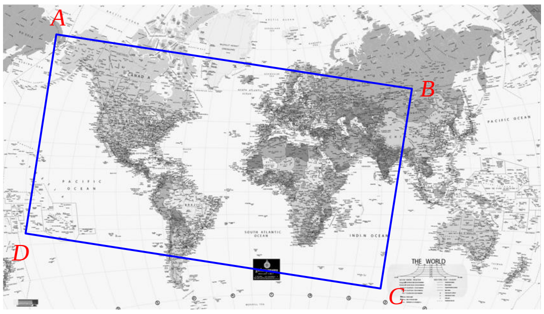


In general, the sides of the two maps are not parallel as shown below.

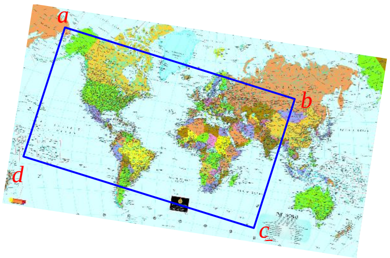


The following context is not a strict mathematical proof, but it is for you to believe that the fact is true.

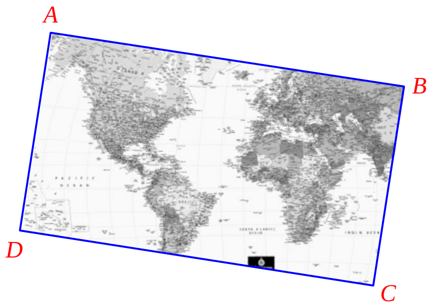
In the figure below rectangle $ABCD$ is the position of the $S$ map on the $L$ map.



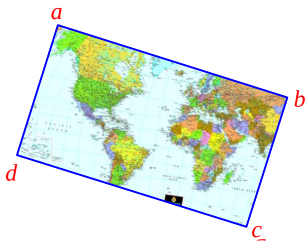

There is the corresponding rectangle on the $S$ map, which is rectangle $abcd$.



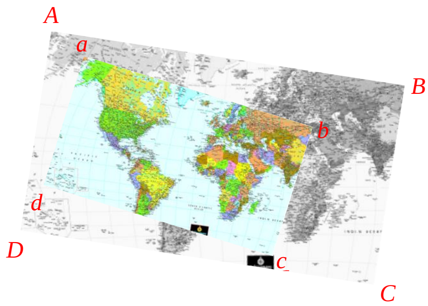

Now we tear off the part of the $L$ map outside rectangle $ABCD$.




We also tear off the corresponding part of the $S$ map outside rectangle $abcd$.

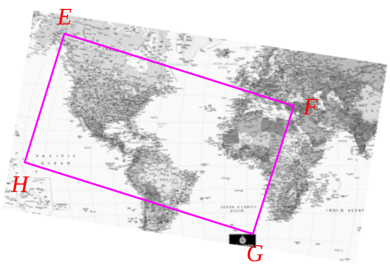



Then we place them together as their original positions.



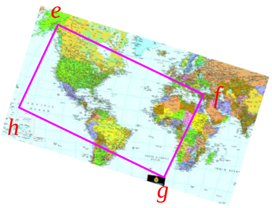

Now we have a smaller $S$ map on a smaller $L$ map. These two smaller maps are the same except their sizes.

Let rectangle $EFGH$ be the position of the smaller $S$ map on the smaller $L$ map.

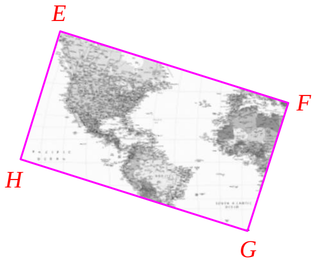


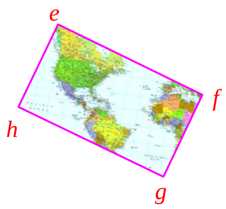
The corresponding rectangle on the smaller *S* map is marked with *efgh*.



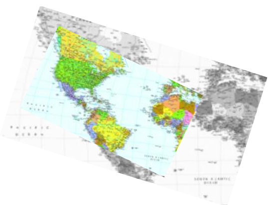

Now we tear off the part of the smaller *L* map outside rectangle *EFGH*.




We also tear off the corresponding part of the smaller *S* map outside rectangle *efgh*.

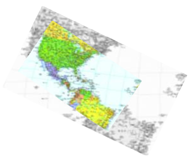



Then we put them together as their original positions again.

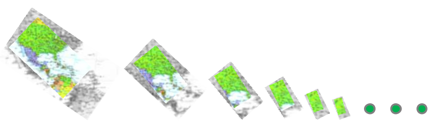


Now we have an even smaller *S* map on a smaller *L* map. These two maps are the same except their sizes.

If we tear off the outside parts of two maps again, the maps become further smaller.



If we do it again and again,



we will have a very small *S* map on a very small *L* map, which are the same except their sizes.

Eventually, the two maps become "points". Because we keep the *S* map and the *L* map always the same, the "points" represent the same place on the two maps.

## Clues to Creative Thinking Problems 49 to 51

### 49. 9 – 1 = 10

This is a tricky question.

### 50. Who Is Taller?

Find somebody who can be compared with *TS* and with *ST*.

### 51. 12 Balls

First weighing: 4 balls against 4 balls.

If the scale is in balance, the bad ball is in the third group of 4 balls. Then it is not difficult to determine the bad ball with two weighings.

If the scale is not in balance, assume that balls 1, 2, 3, and 4 are heavier than balls 5, 6, 7, and 8.

Second weighing: balls 1, 2, 5 against balls 3, 4, 6.

Now go ahead.

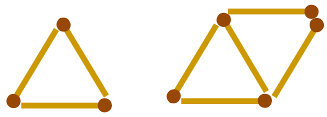
## Creative Thinking Problems 52 to 54

### 52. Make One Word

Rearrange "new door" to make one word instead.

### 53. Make 4 Equilateral Triangles

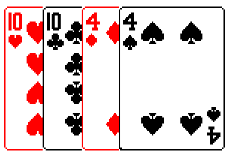
With 3 matchsticks we can make one equilateral triangle. With five we can make two equilateral triangles.



Make four equilateral triangles with six matchsticks without bending and breaking any matchstick.

### 54. Another Challenge to Make 24

Make 24 with



See the rules in Creative Thinking Problem 6 appearing in *Issue 2, Volume 1*.

(Clues and solutions will be given in the next issues.)