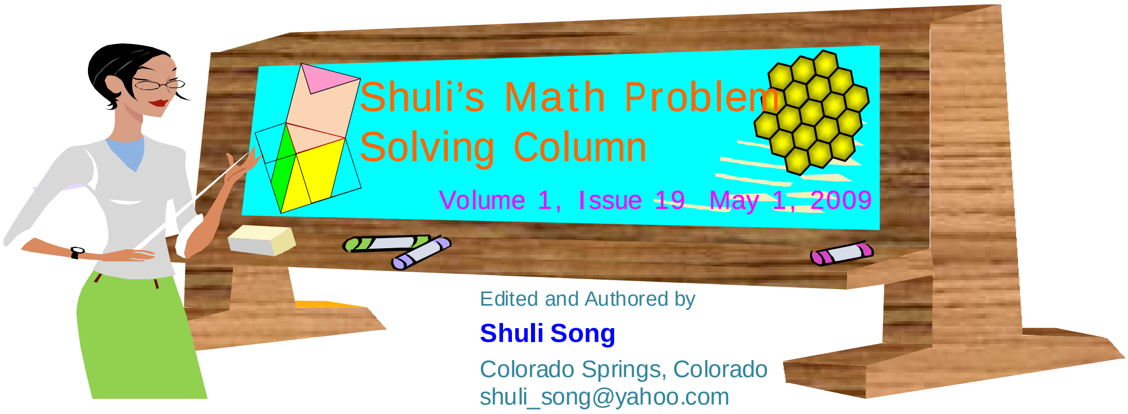


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## Math Trick

### Mental Calculation: $\overline{199a}\times\overline{199b}$

### The Trick

Mentally calculate:

$1997\times1998=$ $1996\times1993=$ $1995\times1992=$

$1999\times1994=$ $1991\times1992=$ $1998\times1995=$

To calculate the product of two numbers close to 2000, we have a short cut.

Write the multiplications in the general form: $\overline{199a}\times\overline{199b}$ where *a* and *b* are digits.

Let $c=2000-\overline{199a}$ and $d=2000-\overline{199b}$ .

Then $\overline{199a}\times\overline{199b}=(2000-c)\times(2000-d)$.

The steps are shown through the following examples.

*Example 1*

Calculate $1993\times1998$.

*Step 1*: Calculate $c=2000-\overline{199a}$ and $d=2000-\overline{199b}$ .

In this example, $c=2000-1993=7$ and $d=2000-1998=2$ .

*Step 2*: Calculate $\overline{199a}-d$ or $\overline{199b}-c$ .

In this example, $\overline{199a}-d=1993-2=1991$ or $\overline{199b}-c=1998-7=1991$.

*Step 3*: Calculate $2\times(\overline{199a}-d)$ or $2\times(\overline{199b}-c)$ .

In this example, $1991\times2=3982$ .

*Step 4*: Calculate $c\times d$ .

In this example, $7\times2=14$ .

*Step 5*: Attach the result in step 4 as three digits to the right of the result in step 3.

In this example, attach 014 to the right of 3982: 3982014.

Now we are done: $1993\times1998=3982014$.

*Example 2*

Calculate $1998\times1997$ .

*Step 1*: Calculate $2000-1998=2$ and $2000-1997=3$.

*Step 2*: Calculate $1998-3=1995$ or $1997-2=1995$.

*Step 3*: Calculate $1995\times2=3990$ .

*Step 4*: Calculate $2\times3=6$ , treated as three digits: 006

*Step 5*: Attach 006 to the right of 3990: 3990006.

We have $1997\times1998=3990006$.

This works for any two numbers close to 2000.

*Example 3*

Calculate $1989\times1993$.

*Step 1*: Calculate $2000-1989=11$ and $2000-1993=7$ .

*Step 2*: Calculate $1989-7=1982$ or $1993-11=1982$.

*Step 3*: Calculate $1982\times2=3964$.

*Step 4*: Calculate $11\times7=77$ , treated as three digits: 077

*Step 5*: Attach 077 to the right of 3964: 3964077.

Then $1989 \times 1993 = 3964077$.

*Example 4*

Calculate $1896 \times 1893$.

*Step 1*: Calculate $2000 - 1896 = 104$ and $2000 - 1893 = 107$.

*Step 2*: Calculate $1896 - 107 = 1789$ or $1893 - 104 = 1789$.

*Step 3*: Calculate $1789 \times 2 = 3578$.

*Step 4*: Calculate $104 \times 107 = 11128$.

Recall how to calculate $\overline{10a} \times \overline{10b}$ mentally.

*Step 5*: Add 11, the first two digits of 11128, to 3578 yielding 3589, and attach 128 to the right of 3589: 3589128.

We obtain $1896 \times 1893 = 3589128$.

## Why Does This Work?

$$\overline{199a} \times \overline{199b} = (2000 - c) \times (2000 - d)$$
$$= 4000000 - 2000c - 2000d + cd$$
$$= 2000(2000 - c - d) + cd = 2000(\overline{199a} - d) + cd$$

or

$$= 2000(\overline{199b} - c) + cd$$

This shows that to calculate $\overline{199a} \times \overline{199b}$, we may do

*Step 1:* Calculate $c = 2000 - \overline{199a}$ and $d = 2000 - \overline{199b}$.

*Step 2:* Calculate $\overline{199a} - d$ or $\overline{199b} - c$.

*Step 3*: Calculate $2 \times (\overline{199a} - d)$ or $2 \times (\overline{199b} - c)$.

*Step 4:* Calculate $c \times d$.

*Step 5:* Attach $c \times d$ as THREE digits to the right of $2 \times (\overline{199a} - d)$ or $2 \times (\overline{199b} - c)$.

## Mental Calculation: $\overline{n99a} \times \overline{n99b}$

The similar procedure applies to the multiplications in the form $\overline{n99a} \times \overline{n99b}$ where $n$ is a digit greater than 2. Instead of 2 we have to multiply $\overline{n99a} - d$ or $\overline{n99b} - c$ by $n+1$ in step 3.

*Example 5*

Calculate $2996 \times 2993$.

*Step 1*: Calculate $3000 - 2996 = 4$ and $3000 - 2993 = 7$.

*Step 2*: Calculate $2993 - 4 = 2989$ or $2996 - 7 = 2989$.

*Step 3*: Calculate $2989 \times 3 = 8967$.

*Step 4*: Calculate $4 \times 7 = 28$, treated as three digits: 028

*Step 5*: Attach 028 to the right of 8967: 8967028.

We obtain $2996 \times 2993 = 8967028$.

*Example 6*

Calculate $3986 \times 3997$.

*Step 1*: Calculate $4000 - 3986 = 14$ and $4000 - 3997 = 3$.

*Step 2*: Calculate $3986 - 3 = 3983$ or $3997 - 14 = 3983$.

*Step 3*: Calculate $3983 \times 4 = 15932$.

*Step 4*: Calculate $14 \times 3 = 42$, treated as three digits: 042

*Step 5*: Attach 042 to the right of 15932: 15932042.

So $3986 \times 3997 = 15932042$.

*Example 7*

Calculate $5987 \times 5989$.

*Step 1*: Calculate $6000 - 5987 = 13$ and $6000 - 5989 = 11$.

*Step 2*: Calculate $5987 - 11 = 5976$ or $5989 - 13 = 5976$.

*Step 3*: Calculate $5976 \times 6 = 35856$.

*Step 4*: Calculate $13 \times 11 = 143$.

*Step 5*: Attach 143 to the right of 35856: 35856143.

Then $5987 \times 5989 = 35856143$.

*Example 8*

Calculate $7895 \times 7893$.

*Step 1*: Calculate $8000 - 7895 = 105$ and $8000 - 7893 = 107$.

*Step 2*: Calculate $7895 - 107 = 7788$ or $7893 - 105 = 7788$.

*Step 3*: Calculate $7788 \times 8 = 62304$.

*Step 4*: Calculate $107 \times 105 = 11235$.

*Step 5*: Add 11, the first two digits of 11235, to 62304 yielding 62315, and attach 235 to the right of 62315: 62315235.

We have $7895 \times 7893 = 62315235$.

## Practice Problems I

| | | |
|---|---|---|
| $2996 \times 2997 =$ | $2991 \times 2998 =$ | $2993 \times 2995 =$ |
| $2991 \times 2996 =$ | $2995 \times 2996 =$ | $2999 \times 2992 =$ |
| $2999^2 =$ | $2997^2 =$ | $2996^2 =$ |
| $2986 \times 2997 =$ | $2991 \times 2987 =$ | $2993 \times 2985 =$ |
| $2987 \times 2984 =$ | $2989 \times 2985 =$ | $2984 \times 2983 =$ |

## Practice Problems II

| | | |
|---|---|---|
| $3996 \times 3995 =$ | $4997 \times 4999 =$ | $5997 \times 5994 =$ |
| $6992 \times 6991 =$ | $7993 \times 7998 =$ | $8993 \times 8996 =$ |
| $3896 \times 3997 =$ | $4891 \times 4989 =$ | $5893 \times 5995 =$ |
| $6892 \times 6891 =$ | $7895 \times 7898 =$ | $8893 \times 8899 =$ |
| $3895 \times 3897 =$ | $4889 \times 4899 =$ | $5894 \times 5893 =$ |
| $6887 \times 6899 =$ | $7885 \times 7884 =$ | $8890 \times 8883 =$ |

## Math Competition Skill

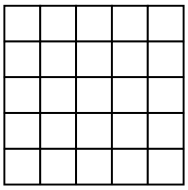
### Counting Squares

Counting squares in grids is quite popular in junior mathematics competitions. This short lesson focuses on how we count squares whose sides are parallel to grid lines.

### Examples

*Example 1*

How many squares of all sizes can you count in the 5 by 5 grid below?



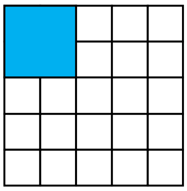
*Answer:* 55

*Solution:*

There are five different squares: one by one squares, two by two squares, three by three squares, four by four squares, and five by five squares.

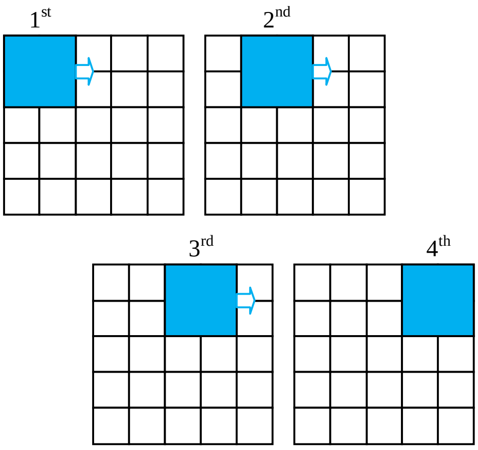
Obviously, there are $5 \times 5 = 25$ one by one squares.

To count two by two squares, we place a two by two square at the left-top corner in the grid.



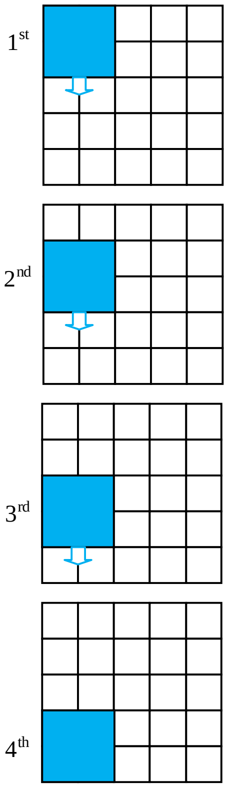
Now we move it.

First we move it horizontally.




There are four positions, including the left-top position, to move the square horizontally.

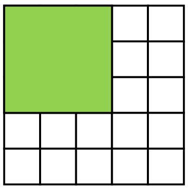
Then we move it vertically.




There are four positions, including the left-top position, to move the square vertically.

So there are $4 \times 4 = 16$ positions to move the square in the grid. That is, there are 16 two by two squares.

To count three by three squares, we also place a three by three square at the left-top corner in the grid.

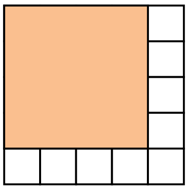


Then we move it.

There are three positions to move it horizontally and three positions to move it vertically.

So there are $3 \times 3 = 9$ positions to move the square in the grid. Hence there are 9 three by three squares.

Similarly, there are $2 \times 2 = 4$ four by four squares.

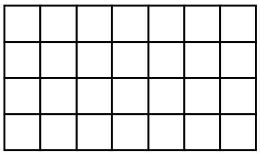


There is only 1 five by five square.

Therefore, there are $25 + 16 + 9 + 4 + 1 = 55$ squares of all sizes.

*Example 2*

How many squares of all sizes can you count in the 4 by 7 grid below?



*Answer:* 60

*Solution:*

There are four different squares: one by one squares, two by two squares, three by three squares, and four by four squares.

There are $4\times 7=28$ one by one squares.

There are $3\times 6=18$ two by two squares.

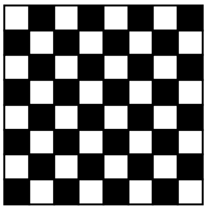
There are $2\times 5=10$ three by three squares.

There are $1\times 4=4$ four by four squares.

Altogether there are $28+18+10+4=60$ squares of all sizes.

*Example 3*

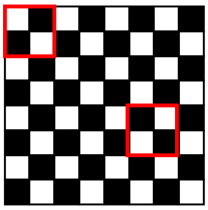
An 8 by 8 checkerboard has alternating black and white squares. How many distinct squares, with sides on the grid lines of the checkerboard (horizontal and vertical) and containing at least 2 black squares and at most 8 black squares, can be drawn on the checkerboard?



*Answer:* 110

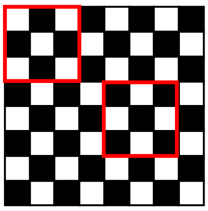
*Solution:*

A two by two square contains 2 black squares.



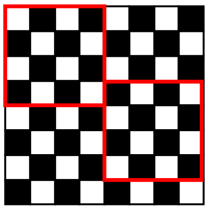
There are $7\times 7=49$ two by two squares.

A three by three square contains either 4 or 5 black squares.



There are $6\times 6=36$ three by three squares.

A four by four square contains 8 black squares.



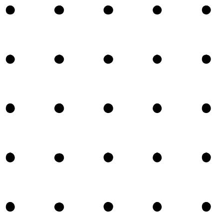
There are $5\times 5=25$ four by four squares.

A five by five or larger square contains more than 8 black squares.

Therefore, the answer is $49+36+25=110$.

*Example 4*

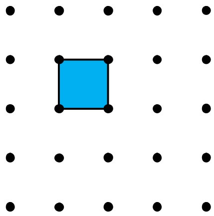
How many squares with horizontal and vertical sides can be formed using points of the grid as vertices?



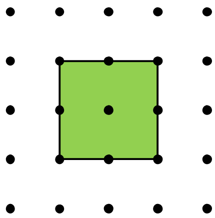
*Answer:* 30

*Solution:*

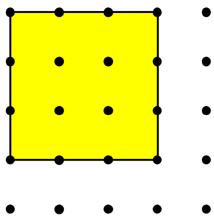
There are $4\times 4=16$ one by one squares, one of which is shown in the figure:



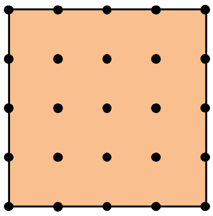
There are $3\times 3=9$ two by two squares, one of which is shown below:



There are $2\times 2=4$ three by three squares, one of which is shown in the figure below:



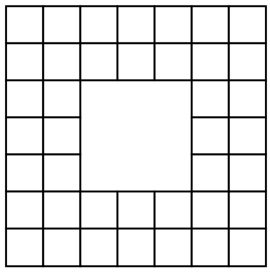
The only 1 four by four square is shown below:



Altogether, there are $16+9+4+1=30$ squares.

*Example 5*

How many squares of all sizes can you count in the 7 by 7 grid if the 9 small squares in the center are removed?

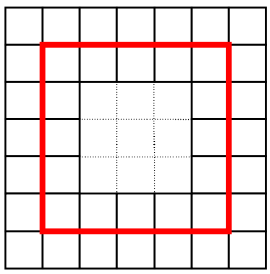


*Answer:* 79

*Solution:*

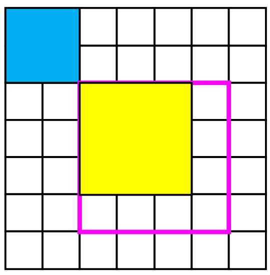
Obviously, there are $49-9=40$ one by one squares.

If the 9 squares in the center are not removed, we have $6\times6=36$ two by two squares. After the 9 squares are removed, all two by two squares in the central $5\times5$ grid shown in the figure below are destroyed. The number of the two by two squares destroyed is $4\times4=16$.



So there are $36-16=20$ two by two squares in the original shape.

We can also find the number of two by two squares by placing a two by two square at the left-top corner and moving it. There are 20 positions to move it.



There is only 1 three by three square (the yellow in the above figure). There are 4 four by four squares, one of which is drawn with pink in the above figure.

There are $3\times3=9$ five by five squares.

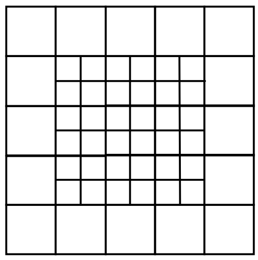
There are $2\times2=4$ six by six squares.

There is only 1 seven by seven square.

Therefore, the answer is $40+20+1+4+9+4+1=79$.

*Example 6*

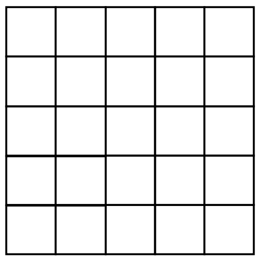
How many squares of all sizes can you count in the figure?



*Answer:* 132

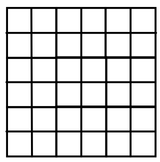
*Solution:*

First we count for the large $5\times5$ grid:



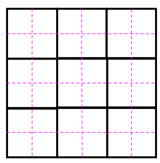
According to *Example* 1 the number of squares of all sizes in this grid is $25+16+9+4+1=55$.

Then we count for the small $6\times6$ grid:



There are $36+25+16+9+4+1=91$ squares of all sizes in this grid.

In the two sets of squares some squares are doubly counted. All the squares in the following $3\times3$ grid are doubly counted:



The number of squares of all sizes in the $3\times3$ grid is

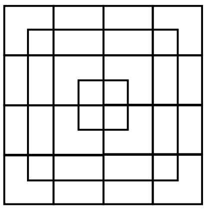
$$9+4+1=14.$$

Therefore, the answer is $55+91-14=132$.

*Example 6*

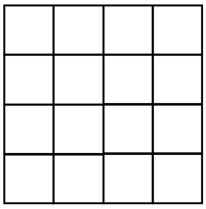
How many squares of all sizes can you count in the figure?



*Answer:* 48

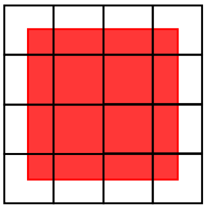
*Solution:*

First we count for the $4 \times 4$ grid:

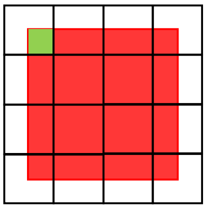


There are $16+9+4+1=30$ squares of all sizes.

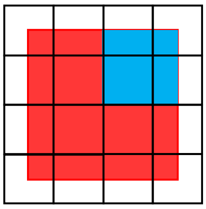
Now we count how many squares are added by adding the red square.



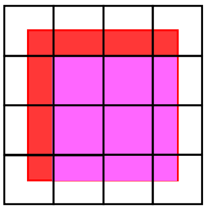
There are four small congruent squares newly formed, one of which is shown in green below.



Four congruent squares (the blue is one) are also formed.



Four congruent squares, one of which is shown in pink, can be seen as well.

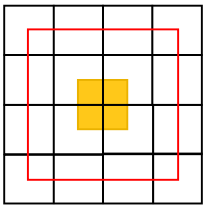


Including the red square itself there are

$$4+4+4+1=13$$

squares thus added.

At last we count how many squares are added by adding the orange square.

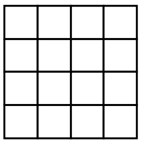


Including the orange square itself 5 squares can be observed.

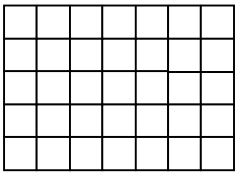
Altogether there are $30+13+5=48$ squares.

## Practice Problems I

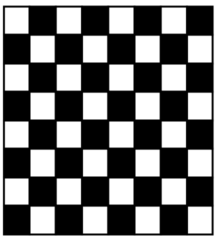
1. How many squares of all sizes can you count in the 4 by 4 grid?



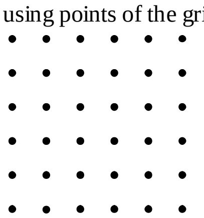
2. How many squares of all sizes can you count in the 5 by 7 grid?



3. An 8 by 8 checkerboard has alternating black and white squares. How many distinct squares, with sides on the grid lines of the checkerboard (horizontal and vertical) and containing at least 5 black squares and at most 12 black squares, can be drawn on the checkerboard?

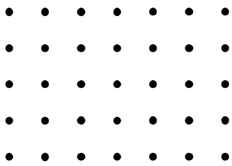


4. How many squares with horizontal and vertical sides can be formed using points of the grid as vertices?

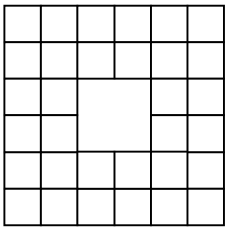


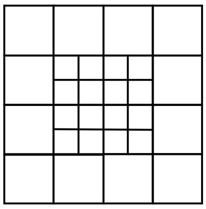
5. How many squares with horizontal and vertical sides can be formed using points of the grid as vertices?



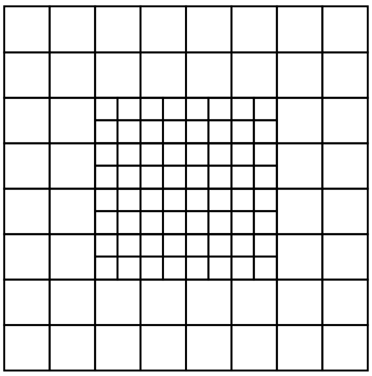
6. How many squares of all sizes can you count in the 6 by 6 grid if the 4 small squares in the center are removed?



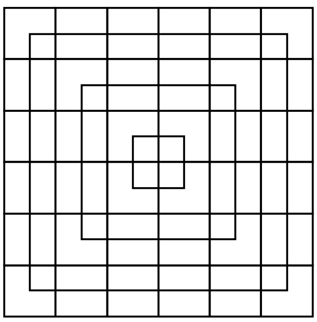
7. How many squares of all sizes can you count in the figure?



8. How many squares of all sizes can you count in the figure?



9. How many squares of all sizes can you count in the figure?

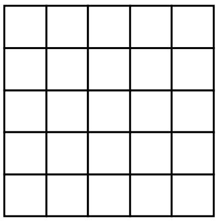


## Practice Problems II

Note: All problems in this section are from MathCounts.

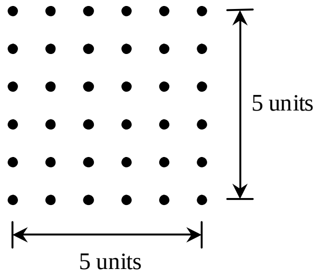
1. (1990 National Team Problem 1)

How many squares are contained in the figure shown?



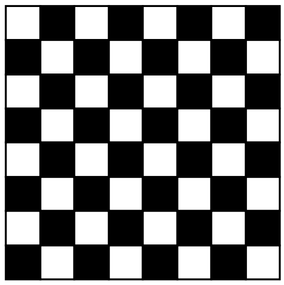
2. (2002 State Countdown Problem 14)

How many squares of area 4 square units have all four vertices on the points in the $6\times 6$ grid of points below?



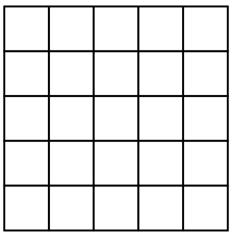

3. (2003 Chapter Team Problem 3)

An 8 by 8 checkerboard has alternating black and white squares. How many distinct squares, with sides on the grid lines of the checkerboard (horizontal and vertical) and containing at least 4 black squares, can be drawn on the checkerboard?



4. (2004 National Countdown Problem 23)

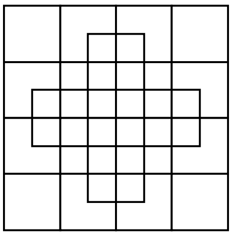
What is the ratio of the number of $2\times 2$ squares to the number of $3\times 3$ squares in the $5\times 5$ square diagram shown (using only the existing horizontal and vertical segments)? Express your answer as a common fraction.



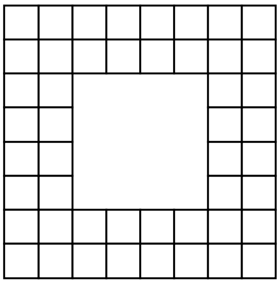
5. (1999 National Sprint Problem 22)

How many squares are pictured?



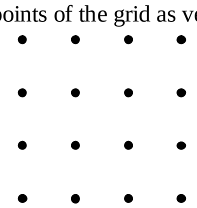
6. (2001-2002 School Handbook Warm-Up 13 Problem 2)

How many squares are determined by the grid lines below if the 48 smaller quadrilaterals are congruent squares?



7. (2005-2006 School Handbook Work-Out 1 Problem 7)

How many squares with horizontal and vertical sides can be formed using points of the grid as vertices?



## A Problem from a Real Math Competition

Today's problem comes from the American Mathematics Competition Grade 8 (AMC8).

(22nd AMC8 2006 Problem 25)

Barry wrote 6 different numbers, one on each side of 3 cards, and laid the cards on a table, as shown. The sums of the two numbers on each of the three cards are equal. The three numbers on the hidden sides are prime numbers. What is the average of the hidden prime numbers?

44 59 38

*Answer:* 14

*Solution:*

Three prime numbers must be different. One odd number and two even numbers are shown. Since all primes except 2 are odd, 2 must be at the other side of 59. Then the same sum is $59+2=61$. Thus the other two primes are $61-44=17$ and $61-38=23$. The average of the three prime numbers is $\frac{2+17+23}{3}=14$.

### Practice Problem

The sum of three primes *a*, *b*, and *c* is 60 where $a<b<c$. Find the possible smallest value of *c*.

## Answers to All Practice Problems in Last Issue

### Mental Calculation

#### Practice Problems I

| | | |
|---|---|---|
| 38612 | 37818 | 37635 |
| 37436 | 38220 | 38208 |
| 37248 | 38412 | 37824 |
| 38214 | 38407 | 38415 |
| 38809 | 38416 | 37636 |

#### Practice Problems II

| | | |
|---|---|---|
| 36642 | 35717 | 35705 |
| 35476 | 36270 | 36288 |
| 34408 | 34968 | 34034 |
| 34404 | 34965 | 33672 |
| 35344 | 34969 | 33489 |

#### Practice Problems III

| | | |
|---|---|---|
| 87912 | 85239 | 83232 |
| 154836 | 152460 | 149382 |
| 243048 | 242526 | 237168 |
| 354614 | 349217 | 485809 |
| 477477 | 633616 | 803710 |

### Systematically Listing According to Shapes

#### Practice Problems I

| | | |
|---|---|---|
| 1. 32 | 2. 72 | 3. 20 |
| 4. 33 | | |

#### Practice Problems II

| | | |
|---|---|---|
| 1. 55 | 2. 38 | 3. 26 |
| 4. 10 | 5. 22 | 3. 67 |
| 7. 36 | 8. 100 | |

### A Problem from a Real Math Competition

$\frac{36}{95}$

## Solutions to Creative Thinking Problems 49 to 51

**49. 9 – 1 = 10**

Taking I (1) away from IX (9), we have X (10) left.

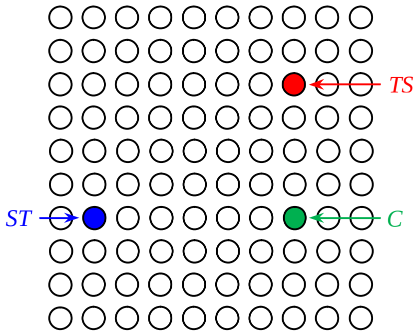
 

## 50. Who Is Taller?

We want to compare *A* with *B*, but we cannot directly compare between them. Very often we may find a third object *C*, which can be compared with both *A* and *B*.

For example, if we find $A > C$ and $B < C$, we know $A > B$.

Look at the $10 \times 10$ matrix below.




Assume the red person is *TS*, and the blue is *ST*.

Now find *C*, who can be compared with both *TS* and *ST*.

We would choose *C*, who is in the same column as *TS*, and in the same row as *ST*. Person *C* is marked with green.

Since *C* is in the same column as *TS*, *TS* is shorter than *C*.

Since *C* is in the same row as *ST*, *ST* is taller than *C*.

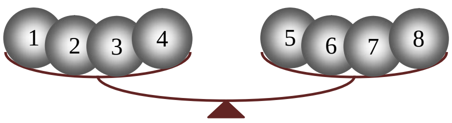
Therefore, *ST* is taller than *TS*.

## 51. 12 Balls

Number the balls from 1 to 12:


Place balls 1 to 4 on the left pan and balls 5 to 8 on the right.

First Weighing

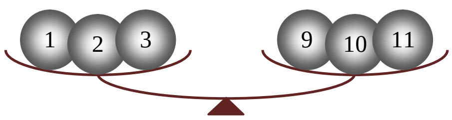



Case 1: The scale is in balance in the first weighing.

The bad ball is in balls 9 to 12. Balls 1 to 8 are all good.

Now place balls 1 to 3 on the left pan and balls 9 to 11 on the right.

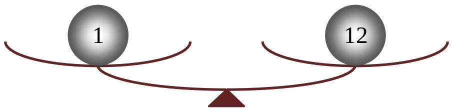
Second Weighing




*Sub-case a: The scale is in balance.*

The bad ball is ball 12. Then place any good ball (say ball 1) on the left pan and ball 12 on the right.

Third Weighing




The scale must not be in balance. With this weighing we can know that the bad ball, ball 12, is lighter or heavier.

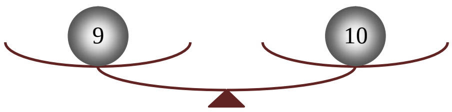
*Sub-case b: The scale is not in balance.*

Then the bad ball is in balls 9 to 11.

Without loss of generality we assume that the left side is heavier. Then the bad ball is lighter.

Now place ball 9 on the left pan and ball 10 on the right.

Third Weighing




If the scale is in balance, the bad ball is ball 11 and ball 11 is lighter.

If the scale is not in balance, whatever is lighter is the bad ball.

Case 2: The scale is not in balance in the first weighing.

Without loss of generality we assume that the left side is heavier.

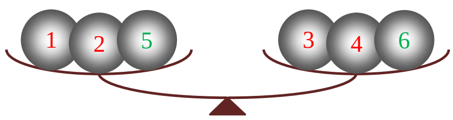
Now we color numbers 1 to 4 with red and numbers 5 to 8 with green. A red number indicates that the corresponding ball is in the heavier side in the first weighing, while a green number means that the corresponding ball is in the lighter side.

We can conclude that

1. If a ball numbered with red is bad, it must be heavier than a good ball.
2. If a ball numbered with green is bad, it must be lighter than a good ball.

Now place balls 1, 2, and 5 on the left pan, and balls 3, 4, and 6 on the right.

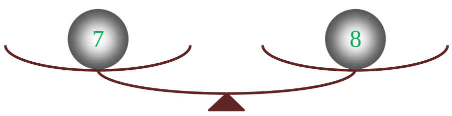
Second Weighing




*Sub-case a: The scale is in balance.*

One of balls 7 and 8 is bad, and the bad ball is lighter. Then place ball 7 on the left pan and ball 8 on the right.

Third Weighing




Whatever is lighter is the bad ball in this weighing.

*Sub-case b: The scale is not in balance.*

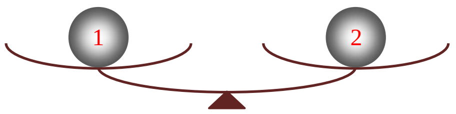
Without loss of generality we assume that the left side is heavier.

Then balls 3, 4, and 5 are all good. Otherwise, if one of balls 3 and 4 is bad, it must be heavier (remember the meaning of the red color) so that the right side is heavier. If ball 5 is bad, it must be lighter (recall the meaning of the green color) so that the left side is lighter.

Now one of balls 1, 2, and 6 is bad.

Place ball 1 on the left pan and ball 2 on the right.

Third Weighing




If the scale is in balance, ball 6 is bad and the bad ball is lighter.

If the scale is not in balance, whatever is heavier is the bad ball.

In any case we can identify the bad ball and determine whether it is lighter or heavier by weighing three times.

## Clues to Creative Thinking Problems 52 to 54

### 52. Make One Word

This is a tricky question.

### 53. Make 4 Equilateral Triangles

Go to *Creative Thinking Problem* 28, the problem, the clue, and the solution of which are presented in *Issues 10*, *11*, and *12*, respectively, *Volume 1*. It may give you a clue.

### 54. Another Challenge to Make 24

$\frac{10\times10}{4}$ is very close to 24.

## Creative Thinking Problems 55 to 57

### 55. 3 x 3 Matrix

See the number table:

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
| 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 |
| 17 | 18 | 19 | 20 | 21 | 22 | 23 | 24 |
| 25 | 26 | 27 | 28 | 29 | 30 | 31 | 32 |
| 33 | 34 | 35 | 36 | 37 | 38 | 39 | 40 |
| 41 | 42 | … | … | … | … | … | … |
| … | … | … | … | … | … | … | … |
| … | … | … | … | … | … | … | … |

Use a rectangle to catch 9 numbers in a $3\times3$ matrix. Can the sum of the nine numbers in the matrix be

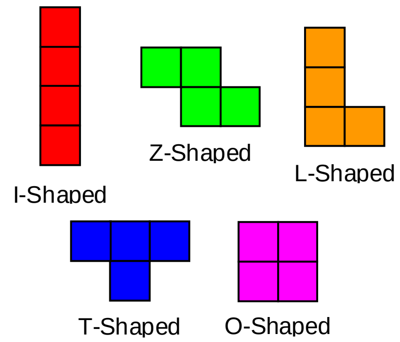
(1) 20090501?

(2) 123456789?

(3) 987654321?

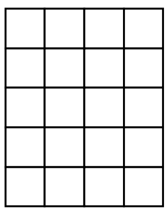
### 56. Covering with Tetrominos

A tetromino is a shape connecting four congruent squares side by side. We consider two shapes to be the same when one can be overlapped with the other by rotation and/or flipping. Then there are five different shapes of tetrominos.




They are called *I-Shaped*, *Z-Shaped*, *L-Shaped*, *T-Shaped*, *and O-Shaped*, respectively, as shown.

Can you fit them together to make the $4\times5$ rectangle shown below? Or say, can you use those five tetrominoes to cover the $4\times5$ rectangle?

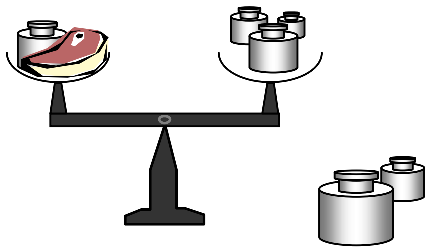


If you can, how? If you cannot, why?

### 57. Weighing Meat II

You have a pan scale to weigh meat up to 100 pounds. Any piece of meat has a weight of whole pounds, that is, 1 pound, 2 pounds, 3 pounds etc. This time you may place meat and weight(s) together on any pan.

What is the least number of weights?



(Clues and solutions will be given in the next issues.)