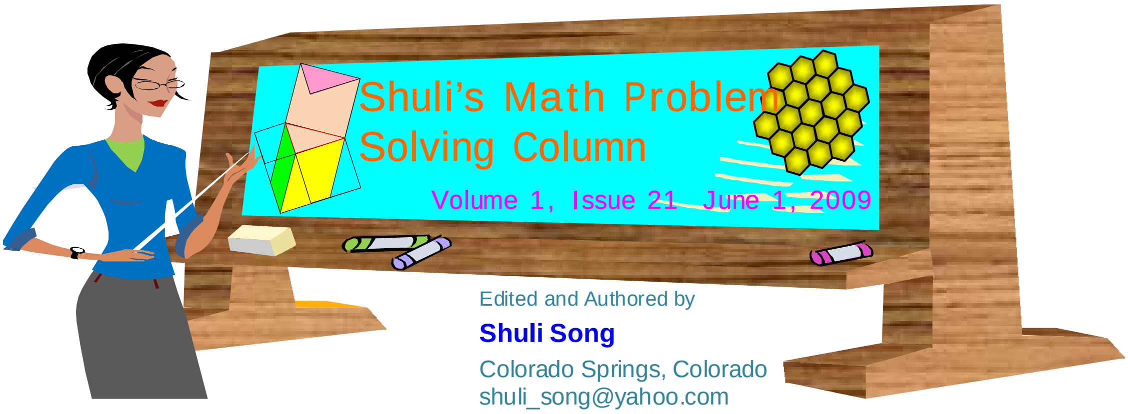

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Contents

Math Trick

Mental Calculation: $\overline{ab}\times\overline{cd}$

The Trick

In the last issue we presented how to calculate the following multiplications mentally:

$26\times32=$ $18\times24=$ $37\times41=$

$47\times52=$ $39\times45=$ $24\times38=$

These multiplications are in the general form: $\overline{ab}\times\overline{cd}$ where a, b, c, and d are digits with $\overline{ab}$ close to $\overline{cd}$.

In this short lesson I will present another short cut through examples.

Example 1

Calculate 18×23.

If we treat 18 as $\overline{2a}$ with $a=-2$, then this is a multiplication in $\overline{2a}\times\overline{2b}$.

Step 1: Calculate $\overline{2a}+b$ or $\overline{2b}+a$.
In this example, $18+3=21$ or $23+(-2)=21$.

Step 2: Multiply the result in step 1 by 2.
In this example, $21\times2=42$.

Step 3: Calculate $a\times b$.
In this example, $-2\times3=-6$.

Step 3 "Add" them this way:

$$\begin{array}{rrrr} & 4 & 2 & \\ - & & & 6 \\ \hline & 4 & 1 & 4 \end{array}$$

We are done: $18\times23=414$.

Example 2

Calculate 24×32.

Treat 24 as $\overline{3a}$ with $a=-6$. Then this is a multiplication in $\overline{3a}\times\overline{3b}$.

Step 1: Calculate $\overline{3a}+b$ or $\overline{3b}+a$.
In this example, $24+2=26$ or $32+(-6)=26$.

Step 2: Multiply the result in step 1 by 3.
In this example, $26\times3=78$.

Step 3: Calculate $a\times b$.
In this example, $-6\times2=-12$.

Step 4 "Add" them:

$$\begin{array}{rrrr} & 7 & 8 & \\ - & & 1 & 2 \\ \hline & 7 & 6 & 8 \end{array}$$

We obtain $24\times32=768$.

Example 3

Calculate 47×51.

Treat 47 as $\overline{5a}$ with $a=-3$. Then this is a multiplication in $\overline{5a}\times\overline{5b}$.

Step 1: Calculate $\overline{5a}+b$ or $\overline{5b}+a$.
In this example, $47+1=48$ or $51+(-3)=48$.

Step 2: Multiply the result in step 1 by 5.
In this example, $48\times5=240$.

Step 3: Calculate $a\times b$.
In this example, $-3\times1=-3$.

Step 4 "Add" them:

$$\begin{array}{rrrr} & 2 & 4 & 0 \\ - & & & 3 \\ \hline 2 & 3 & 9 & 7 \end{array}$$

We obtain $47 \times 51 = 2397$.

Example 4

Calculate 68×74.

Treat it as a multiplication in $\overline{7a} \times \overline{7b}$.

Step 1: Calculate $\overline{7a} + b$ or $\overline{7b} + a$.

In this example, $68 + 4 = 72$ or $74 + (-2) = 72$.

Step 2: Multiply the result in step 1 by 7.

In this example, $72 \times 7 = 504$.

Step 3: Calculate $a \times b$.

In this example, $(-2) \times 4 = -8$.

Step 4 "Add" them:

$$\begin{array}{rrrr} & 5 & 0 & 4 \\ - & & & 8 \\ \hline 5 & 0 & 3 & 2 \end{array}$$

We get $68 \times 74 = 5032$.

Practice Problems

$19 \times 23 =$	$27 \times 16 =$	$17 \times 28 =$
$32 \times 26 =$	$24 \times 36 =$	$28 \times 37 =$
$34 \times 41 =$	$45 \times 38 =$	$43 \times 32 =$
$48 \times 53 =$	$67 \times 54 =$	$42 \times 56 =$
$63 \times 51 =$	$68 \times 73 =$	$74 \times 64 =$

Math Competition Skill

Divisibility by 11

Alternate Digit Difference

Definitions:

The 1st, 3rd, 5th, … digits counted from the right are called *odd placed digits*. The sum of all these digits is called the *odd placed digit sum*. The 2nd, 4th, 6th, … digits are called *even placed digits*. The sum of all these digits is called the *even placed digit sum*. Subtract the even placed digit sum from the odd placed digit sum. The result is called the *alternate digit difference*.

Example 1

What is the alternate digit difference of 3728?

Answer: 10

Solution:

The odd placed digit sum is $8 + 7 = 15$. The even placed digit sum is $2 + 3 = 5$. Then the alternate digit difference is $15 - 5 = 10$.

Example 2

What is the alternate digit difference of 9876543210?

Answer: –5

Solution:

The odd placed digit sum is $0 + 2 + 4 + 6 + 8 = 20$. The even placed digit sum is $1 + 3 + 5 + 7 + 9 = 25$. Then the alternate digit difference is $20 - 25 = -5$.

Divisibility by 11

We have the following theorem for divisibility by 11.

Theorem

A number is divisible by 11 if and only if the alternate digit difference of the number is divisible by 11.

Note that 0 is divisible by any natural number.

Example 3

Is 47,839 divisible 11?

Answer: Yes.

Solution:

The odd placed digit sum is $9 + 8 + 4 = 21$, and the even placed digit sum is $3 + 7 = 10$. Then the alternate digit difference is $21 - 10 = 11$, which is divisible by 11. So 47,839 is divisible by 11.

Example 4

Is 123456789 divisible 11?

Answer: No.

Solution:

The odd placed digit sum is $9 + 7 + 5 + 3 + 1 = 25$, and the even placed digit sum is $8 + 6 + 4 + 2 = 20$. Then the alternate digit difference is $25 - 20 = 5$, which is not divisible by 11. So 123456789 is not divisible by 11.

Example 5

Is 21728190 divisible 11?

Answer: Yes.

Solution:

The odd placed digit sum is $0 + 1 + 2 + 1 = 4$, and the even placed digit sum is $9 + 8 + 7 + 2 = 26$. Then the alternate digit difference is $4 - 26 = -22$ which is divisible by 11. So 21728190 is divisible by 11.

Proof of the Theorem

Let N be an $(n+1)$-digit number $\overline{a_n a_{n-1} \cdots a_1 a_0}$. Assume n is odd without loss of generality.

Express $N = \overline{a_n a_{n-1} \cdots a_1 a_0}$ in the base 10 expansion:

$$\begin{aligned} \overline{a_n a_{n-1} \cdots a_1 a_0} &= a_n \times 10^n + a_{n-1} \times 10^{n-1} + \cdots + a_1 \times 10 + a_0 \\ &= a_n \times 1\overbrace{000 \cdots 0}^{n-1\ 0's}1 + a_{n-1} \times \overbrace{999 \cdots 9}^{n-1\ 9's} + \cdots + a_2 \times 99 + a_1 \times 11 \\ &\quad - a_n + a_{n-1} - a_{n-2} + a_{n-3} + \cdots + a_2 - a_1 + a_0. \end{aligned}$$

Note that $\overbrace{1000\cdots01}^{|\leftarrow m\ 0's\rightarrow|}$ and $\overbrace{999\cdots9}^{|\leftarrow m\ 9's\rightarrow|}$ where m is even are always divisible by 11.

Therefore, $N=\overline{a_n a_{n-1}\cdots a_1 a_0}$ is divisible by 11 if and only if $-a_n+a_{n-1}-a_{n-1}+a_{n-3}+\cdots+a_2-a_1+a_0$, which is the alternate digit difference

$$\left(a_{n-1}+a_{n-3}+\cdots+a_2+a_0\right)-\left(a_n+a_{n-2}+\cdots+a_3+a_1\right),$$

is divisible by 11.

Remainder upon Division by 11

For a number, calculate the alternate digit difference.

If the result is more than or equal to 11, calculate the alternate digit difference again. Or subtract 11 from it. If the result is still more than or equal to 11, subtract 11 again until the result is less than 11. If the alternate digit difference is less than 0, add 11 to it. If the result is still less than 0, add 11 again until the result is larger than or equal to 0. The final result is the remainder of the number upon division by 11.

Example 6

What is the remainder of 9876543210 upon division by 11?

Answer: 6

Solution:

From *example 1*, the alternate digit difference is -5. $-5+11=6$. Then 6 is the remainder.

Example 7

What is the remainder of 6372819 upon division by 11?

Answer: 2

Solution:

The odd placed digit sum is $9+8+7+6=30$, and the even placed digit sum is $1+2+3=6$. The alternate digit difference is $30-6=24$. $24-11=13$. $13-11=2$. Then 2 is the answer.

Examples of Problem Solving

Example 8

Six-digit number $\overline{1072m3}$ is divisible by 11. Find m.

Answer: 8

Solution:

The odd placed digit sum is $3+2+0=5$, and the even placed digit sum is $m+7+1=8+m$. The alternate digit difference is $5-(8+m)=-3-m$. The only value of digit m is 8 such that $-3-m$ is divisible by 11.

Example 9

Five-digit number $\overline{m49n7}$ is divisible by 11 where m and n are digits with $m>0$. How many different pairs of values for m and n are there?

Answer: 8

Solution:

The odd placed digit sum is $m+16$, and the even placed digit sum is $n+4$. The alternate digit difference is $m-n+12$. Then $m-n+12$ is divisible by 11. So $m-n=-1$. m could be from 1 to 8, and n from 2 to 9 respectively. Therefore, there are 8 different pairs of values for m and n.

Example 10

$20!=\overline{ab32902008176640000}$ where a and b are digits. Find a and b.

Answer: $a=2$ and $b=4$.

Solution:

20! is divisible by 9. Then the digit sum $a+b+48$ is divisible by 9. So $a+b=6$ or $a+b=15$.

20! is divisible by 11. The odd placed digit sum is $25+a$, and the even placed digit sum is $23+b$. The alternate digit difference is $2+a-b$. Then $2+a-b$ is divisible by 11. So $a-b=-2$ or $a-b=9$.

Solving for a and b we have $a=2$ and $b=4$.

Example 11

Prove that any six-digit number $\overline{abc,abc}$ is divisible by 11 where a, b, and c are digits with $a>0$.

Proof:

The odd placed digit sum is $c+a+b$, and the even placed digit sum is $b+c+a$. The alternate digit difference is 0, which is divisible by 11. Therefore, $\overline{abc,abc}$ is divisible by 11.

Practice Problems

1. Circle the numbers divisible by 11:

   234 121 213,532 1331 123321 777 26189 456456 7272 2750 13876

2. Find the remainder for each upon division by 11:

   2468 13579 123456 9876 918273645 36843 1485 226754 444 8074231

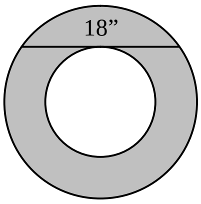
3. Is 6727499949325600009201 divisible by 11?
4. When 931322574615478515625 is divided by 11, what is the remainder?
5. Eight-digit number $\overline{10m02372}$ is divisible by 11. Find m.
6. $29!=\overline{8841761993a39b01954543616000000}$ where a and b are digits. Find a and b.
7. Five-digit number $\overline{2mn48}$ is divisible by 11 where m and n are digits. How many different pairs of values for m and n are there?
8. Prove that any six-digit number $\overline{abccba}$ is divisible by 11 where a, b, and c are digits with $a>0$.

A Problem from a Real Math Competition

Today's problem comes from MathCounts. The problem or a similar problem appeared in MathCounts and other math competitions occasionally.

(MathCounts 1995 National Sprint Problem 17)

A chord of the larger of two concentric circles is tangent to the smaller circle and measure 18 inches. Find the number of square inches in the area of the shaded region.

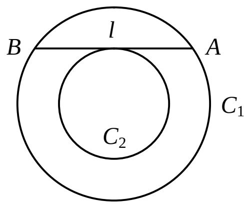


Answer: 81π

Solution:

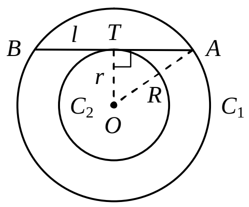
Theorem

In the figure below, C_1 and C_2 are two concentric circles. Chord *AB* of circle C_1 is tangent to circle C_2. Let *l* be the length of *AB*. Then the area of the ring between the two circles are solely determined by *l*, independent of the sizes of the two circles, provided that *l* is fixed.



Proof of the Theorem:

Let *T* be the tangent point, and *O* be the center of the two circles. Let *R* and *r* be the radii of circles C_1 and C_2 respectively.



Draw *OA* and *OT*. Then $OT \perp AB$ and *T* is the midpoint of *AB*. Obviously, $AT = \frac{l}{2}$, $OT = r$ and $OA = R$.

In right ΔOTA $r^2 + \left(\frac{l}{4}\right)^2 = R^2$. So $R^2 - r^2 = \frac{l^2}{4}$.

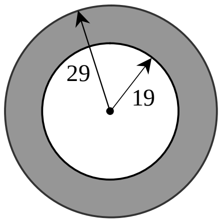
Then the area of the ring is $R^2\pi - r^2\pi = \frac{l^2\pi}{4}$, which is determined by *l* solely.

With $l = 18$ the answer to the problem is $\frac{18^2\pi}{4} = 81\pi$.

Practice Problems

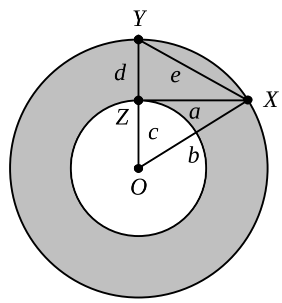
1. (MathCount 2007 National Target Problem 7)

Two concentric circles with radii of 19 and 29 units bound a shaded region. A third circle will be drawn with area equal to that the shaded area. What must the radius of the third circle be? Express your answer in simplest radical form.



2. (5th AMC10B 2004 Problem 10 and 55th AMC 12B 2004 Problem 10)

An annulus is the region between two concentric circles. The concentric circles in the figure have radii *b* and *c*. Let *OX* be a radius of the larger circle, let *XZ* be tangent to the smaller circle at *Z*, and let *OY* be the radius of the larger circle that contains *Z*. Let $a = XZ$, $y = YZ$, and $e = XY$. What is the area of the annulus?



A) πa^2 B) πb^2 C) πc^2 D) πd^2 E) πe^2

3. (20th AHSME 1969 Problem 6)

The area of the ring between two concentric circles is $12\frac{1}{2}\pi$ square inches. The length of a chord of the larger circle tangent to the smaller circle, in inches, is

A) $\frac{5}{\sqrt{2}}$ B) 5 C) $5\sqrt{2}$ D) 10 E) $10\sqrt{2}$

4. (60th AMC12A 2009 Problem 19)

Andrea inscribed a circle inside a regular pentagon, circumscribed a circle around the pentagon, and calculated the area of the region between the two circles. Bethany did the same with a regular heptagon (7 sides). The areas of the two regions were *A* and *B*, respectively. Each polygon had a side length of 2. Which of the following is true?

A) $A = \frac{25}{49}B$ B) $A = \frac{5}{7}B$ C) $A = B$

D) $A = \frac{7}{5}B$ E) $A = \frac{49}{25}B$

Answers to All Practice Problems in Last Issue

Mental Calculation

437	432	476
832	864	1036
1394	1710	1376
2544	3618	2352
3213	4964	4736

Divisibility by 9

1. The numbers divisible by 9 are:

   234 23301 21,213 60606 10119

2. 5 1 6 2 1 3 7 6 8 0

3. Yes 4. 4 5. 9 6. 7

7. 3 8. 17 9. 5

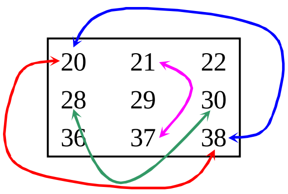
A Problem from a Real Math Competition

6

Solutions to Creative Thinking Problems 55 to 57

55. 3 x 3 Matrix

Look at the matrix. If we pair the numbers as shown, we know that the sum is 9 times the central number.



Two numbers in each pair has a sum of 58 =2 x 29.

Therefore, the sum must be a multiple of 9.

20090501 is not a multiple of 9. So the sum cannot be 20090501.

However, if a number is a multiple of 9, it is not necessarily the sum.

For example, 144 is a multiple of 9, but it cannot be the sum of the nine numbers in a matrix.

The reason follows.

If 144 is the sum of the nine numbers in a matrix, the central number must be $\frac{144}{9} = 16$. However, 16 is in the rightmost column.

If a number is in the leftmost column, it cannot be the central number of a matrix either.

Both 123456789 and 987654321 are multiples of 9.

If 987654321 is the sum of the nine numbers in a matrix, the central number must be $\frac{987654321}{9} = 109739369$.

Recall the shortcut for the divisibility by 8 in *Issue* 6, *Volume* 1. 109739369 yields a remainder of 1 upon division by 8. So it is in the leftmost column. Therefore, 123456789 cannot be the sum.

If 123456789 is the sum of the nine numbers in a matrix, the central number must be $\frac{123456789}{9} = 13717421$.

Since $13717421 = 3 \bmod 8$, the central number is at the third column from the left.

Therefore, 123456789 can be the sum of the nine numbers in a matrix.

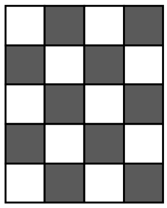
56. Covering with Tetrominos

You may have got "no" as the answer if you have tried. It is correct. But are you able to explain why you cannot?

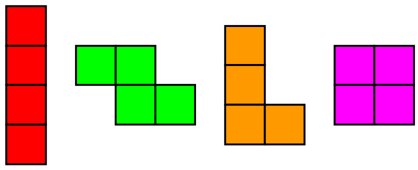
I gave the problem to a child once. He had tried a little while, and then he said "I know the answer is "no", and I feel that the T-shaped is strange."

Yes, the T-shaped is the odd man. Why is it?

Color the board with the standard chessboard coloring:

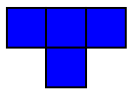


Then any one of the following four tetrominoes covers two black and two white squares:



There are 10 black and 10 white squares. Wherever you place these four shapes on the 4×5 rectangle, two $(10 - 2 \times 4 = 2)$ black and two white squares will be left uncovered.

These four uncovered squares (two black and two white) cannot be covered by



because the T-shaped tetromino covers three black and one white squares, or one black and three white squares only.

57. Weighing Meat II

Again, let us study from small numbers.

We must have a weight of 1 pound for a piece of meat of 1 pound.

As we talked in the "*Weigh Meat I*" problem (*Issue* 15, *Volume* 1), we don't want to make another weight of 1 pound to weigh a piece of meat of 2 pounds. Instead we would like to make a heavier weight.

A weight of 2 pounds works.

However, we may place a piece of meat and weights on one pan.

If we place a piece of meat of 2 pounds with the existing weight of 1 pound together, we would like to make a weight of 3 pounds to balance them.

With the weight of 3 pounds, we can weigh a piece of meat of 3 pounds.

We can weigh a piece of meat of 4 pounds by combining the two existing weights.

To weigh a piece of meat of 5 pounds, we need a new weight.

Since we may place the meat of 5 pounds with the two existing weights (1 pound and 3 pounds) together on one pan, we would like to make a weight of 9 pounds to balance them.

Then we can use the 9-pound weight to balance a piece of meat of 6 pounds and the 3-pound weight.

Combine the 9-pound weight and the 1-pound weight to balance a piece of meat of 7 pounds and the 3-pound weight.

Use the 9-pound weight to balance a piece of meat of 8 pounds and the 1-pound weight.

Using the 9-pound weight we can weigh a piece of meat of 9 pounds.

Combining the 9-pound weight and the 1-pound weight we can weigh a piece of meat of 10 pounds.

Combine the 9-pound weight and the 3-pound weight to balance a piece of meat of 11 pounds and the 1-pound weight.

Combining the 9-pound weight and the 3-pound weight we can weigh a piece of meat of 12 pounds.

Using all three existing weights we can weigh a piece of meat of 13 pounds.

To weigh a piece of meat of 14 pounds, we need a new weight.

Similarly we would make a weight of 27 pounds to balance the meat of 14 pounds and all three existing weights (1-pound, 3-pound, and 9-pound).

Now we have the pattern: the weights are powers of 3.

So the fifth weight is 81 pounds.

Five weights are enough, which are 1 pound, 3 pounds, 9 pounds, 27 pounds, and 81 pounds respectively.

In fact, with these five weights we can weigh a piece of meat of up to $1+3+9+27+81=121$ pounds.

To weigh a piece of meat of up to 100 pounds, you may have a different set of five weights. One set may be five weights of 1 pound, 3 pounds, 9 pounds, 27 pounds, and 60 pounds respectively.

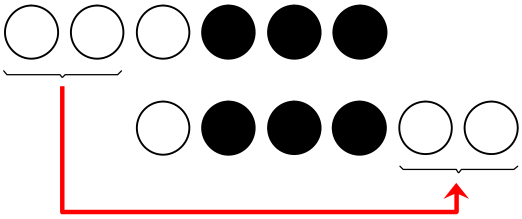
Clues to Creative Thinking Problems 58 to 60

58. A Division

Factorize 452.

59. Moving Checkers

Let me tell you the first move:



First move

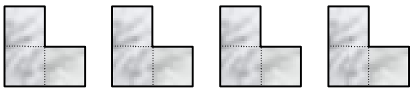
60. A Checkerboard Game

For a game strategy, working backwards very often helps.

Creative Thinking Problems 61 to 63

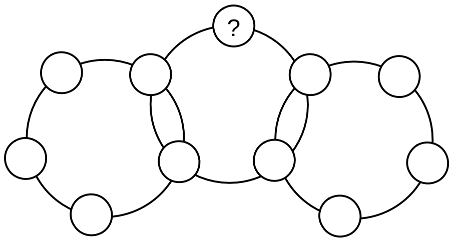
61. Four Trominos to One

Arrange the four L-shaped trominos together to make a larger shape similar to them.



62. Magic Circles

Fill 0 to 10 into the 11 small circles such that the five numbers on each of the three large circles have a sum of 24. Which number must replace the question mark?



63. Two Smart Students of Dr. Math

Dr. Math has two smart students Al and Bob. Dr. Math picks up two integers m and n from 1 to 9 with $m \le n$. Dr. Math calculates the sum of m and n and tells Al the sum, and calculates the product of m and n and tells Bob the product. Then Al and Bob have the following conversations:

Al: I don't know what m and n are, but I'm sure you don't know either.

Bob: Now I know what m and n are.

What does Dr. Math tell Al?

(Clues and solutions will be given in the next issues.)