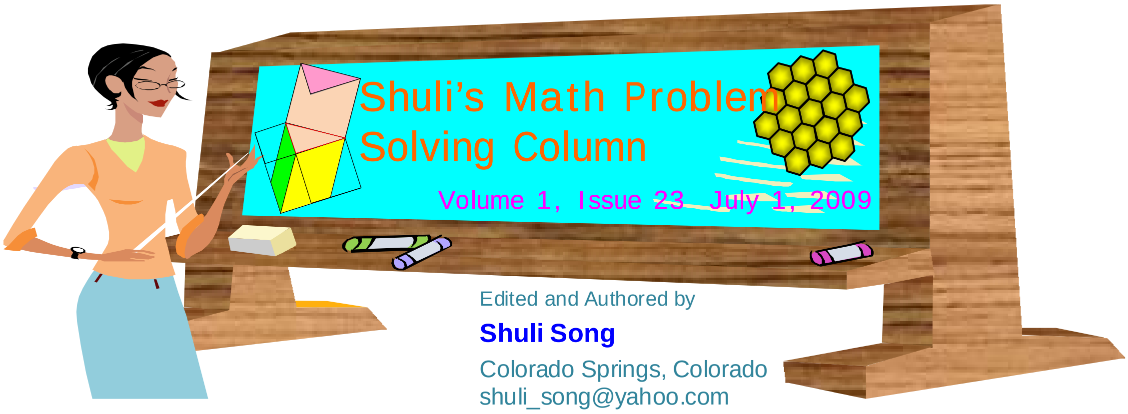

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Contents

Math Trick

Mental Calculation: $\overline{a9} \times \overline{b9}$

The Trick

Mentally calculate:

$79 \times 89 =$	$69 \times 39 =$	$59 \times 29 =$
$99 \times 49 =$	$19 \times 29 =$	$89 \times 59 =$

To calculate the product of two numbers ending in 9, we have a short cut.

Write the multiplications in the general form: $\overline{a9} \times \overline{b9}$ where a and b are digits.

The steps are shown through the following examples.

Example 1

Calculate 39×69.

Step 1: Calculate $(a+1)(b+1)$.

In this example, $(a+1)(b+1) = 4 \times 7 = 28$.

Step 2: Calculate $(a+1)+(b+1)$.

In this example, $(a+1)+(b+1) = 4+7 = 11$.

Step 3: "Subtract" $(a+1)+(b+1)$ from $(a+1)(b+1)$:

$$\begin{array}{rccc} & 2 & 8 & \\ - & & 1 & 1 \\ \hline & 2 & 6 & 9 \end{array}$$

Step 4: Attach 1 to the right of the result in step 3.

In this example, attach 1 to the right of 269: 2691.

Now we are done: $39 \times 69 = 2691$.

Example 2

Calculate 79×89.

Step 1: Calculate $8 \times 9 = 72$.

Step 2: Calculate $8 + 9 = 17$.

Step 3: "Subtract":

$$\begin{array}{rccc} & 7 & 2 & \\ - & & 1 & 7 \\ \hline & 7 & 0 & 3 \end{array}$$

Step 4: Attach 1 to the right of 703: 7031.

We have $79 \times 89 = 7031$.

Example 3

Calculate 29×49.

Step 1: Calculate $3 \times 5 = 15$.

Step 2: Calculate $3 + 5 = 8$.

Step 3: "Subtract":

$$\begin{array}{rccc} & 1 & 5 & \\ - & & & 8 \\ \hline & 1 & 4 & 2 \end{array}$$

Step 4: Attach 1 to the right of 142: 1421.

We obtain $29 \times 49 = 1421$.

Why Does This Work?

$$\begin{aligned} \overline{a9} \times \overline{b9} &= [10(a+1)-1] \times [10(b+1)-1] \\ &= 100(a+1)(b+1) - 10(a+1) - 10(b+1) + 1 \\ &= 10\{10(a+1)(b+1) - [(a+1)+(b+1)]\} + 1 \end{aligned}$$

The above expression suggests the procedure shown in the above examples.

Practice Problems

$69\times 79=$	$19\times 89=$	$39\times 59=$
$69\times 19=$	$59\times 69=$	$99\times 29=$
$29\times 49=$	$49\times 89=$	$79\times 29=$
$89\times 39=$	$39\times 99=$	$59\times 79=$
$79^2=$	$69^2=$	$49^2=$

Math Competition Skill

Divisibility by 99 and 999

Divisibility by 9 and 11

We have learnt how to test whether a number is divisible by 9 and 11. Note that $99=9\times 11$, and 9 and 11 are relatively prime. A number is divisible by 99 if and only if the number is divisible by both 9 and 11.

Example 1

Is 45144 divisible by 99?

Answer: Yes

Solution:

The digit sum of 45144 is

$$4+5+1+4+4=18.$$

Since 18 is divisible by 9, 45144 is divisible by 9.

The alternate digit difference of 45144 is

$$(4+1+4)-(4+5)=0.$$

Since 0 is divisible by 11, 45144 is divisible by 11.

Therefore, 45144 is divisible by 99.

Example 2

Is 111222333 divisible by 99?

Answer: No

Solution:

The digit sum of 111222333 is

$$1+1+1+2+2+2+3+3+3=18.$$

Since 18 is divisible by 9, 111222333 is divisible by 9.

The alternate digit difference of 111222333 is

$$(3+3+2+1+1)-(3+2+2+1)=2.$$

Since 2 is not divisible by 11, 111222333 is not divisible by 11.

Therefore, 111222333 is not divisible by 99.

Example 3

Is 1072841 divisible by 99?

Answer: No

Solution:

The digit sum of 1072841 is

$$1+0+7+2+8+4+1=23.$$

23 is not divisible by 9. So 1072841 is not divisible by 9.

Therefore, 1072841 is not divisible by 99.

Two-Digit Sum and Final Two-Digit Sum

From the right we divide a number into groups each of which has two digits. There may be only one digit in the leftmost group. For example, we divide 123456789 into 5 groups as $1'23'45'67'89$. The *two-digit sum* is defined as the sum of the two-digit numbers in all groups (possibly the one-digit number in the leftmost group). For example, the two-digit sum of 123456789 is

$$1+23+45+67+89=225.$$

If the sum has three or more digits, we calculate the two-digit sum again until we get the result with fewer than three digits. This result is called the *final two-digit sum*.

The final two-digit sum of 123456789 is $2+25=27$.

Theorems for Divisibility by 99

For divisibility by 99, we have the following theorem:

Theorem One

A number is divisible by 99 if and only if the two-digit sum of the number is divisible by 99.

Proof:

Let N be an $(n+1)$-digit number $\overline{a_na_{n-1}\cdots a_3a_2a_1a_0}$. Assume n is odd without loss of generality. Then

$$\begin{aligned} N &= \overline{a_na_{n-1}\cdots a_3a_2a_1a_0} \\ &= \overline{a_na_{n-1}}\times 10^{n-1}+\overline{a_{n-2}a_{n-3}}\times 10^{n-3}+\cdots+\overline{a_3a_2}\times 100+\overline{a_1a_0} \\ &= \overline{a_na_{n-1}}\times \overbrace{999\cdots 9}^{n-1\ 9\text{'s}}+\overline{a_{n-2}a_{n-3}}\times \overbrace{999\cdots 9}^{n-3\ 9\text{'s}}+\cdots+\overline{a_3a_2}\times 99 \\ &\quad +\overline{a_na_{n-1}}+\overline{a_{n-2}a_{n-3}}+\cdots+\overline{a_3a_2}+\overline{a_1a_0}. \end{aligned}$$

Note that $\overline{ab}\times \overbrace{999\cdots 9}^{m\ 9\text{'s}}$ where m is even is always divisible by 99.

So $\overline{a_na_{n-1}}\times \overbrace{999\cdots 9}^{n-1\ 9\text{'s}}+\overline{a_{n-2}a_{n-3}}\times \overbrace{999\cdots 9}^{n-3\ 9\text{'s}}+\cdots+\overline{a_3a_2}\times 99$ is divisible by 99.

Therefore, $N=\overline{a_na_{n-1}\cdots a_3a_2a_1a_0}$ is divisible by 99 if and only if $\overline{a_na_{n-1}}+\overline{a_{n-2}a_{n-3}}+\cdots+\overline{a_3a_2}+\overline{a_1a_0}$, which is the two-digit sum of N, is divisible by 99.

Example 4

Is 93918330 divisible by 99?

Answer: Yes

Solution:

The two-digit sum of 93918330 is

$$93+91+83+30=297.$$

Since 297 is divisible by 99, 93918330 is divisible by 99.

Example 5

Is 1965842 divisible by 99?

Answer: No

Solution:

The two-digit sum of 1965842 is

$$1+96+58+42=197.$$

Since 197 is not divisible by 99, 1965842 is not divisible by 99.

Since we can repeat using theorem one, we have the second theorem:

Theorem Two

A number is divisible by 99 if and only if the final two-digit sum is 99.

Example 6

Is 123456789 divisible by 99?

Answer: No

Solution:

The final two-digit sum of 123456789 is 27.

So 123456789 is not divisible by 99.

Example 7

Is 8783969832 divisible by 99?

Answer: Yes

Solution:

The two-digit sum of 8783969832 is

$$87+83+96+98+32=396.$$

The final two-digit sum is $3+96=99$.

So 8783969832 is divisible by 99.

Remainder upon Division by 99

To find the remainder of a number upon division by 99 we have the following two theorems.

Theorem Three

A number and its two-digit sum have the same remainder upon division by 99.

Theorem Four

The final two-digit sum of a number is the remainder when the number is divided by 99.

Example 8

What is the remainder when 9753186420 is divided by 99?

Answer: 54

Solution:

The two-digit sum of 9753186420 is

$$97+53+18+64+20=252.$$

$252-2\times 99=54$. Or $2+52=54$.

Therefore, 54 is the remainder.

Math Problem Solving

Now we will use the theorems to solve more complicated problems.

Example 9

Eight-digit number $\overline{3ab16800}$ is divisible by 99 where a and b are digits. Find all possible values for $a\cdot b$.

Answer: 0 and 81.

Solution 1:

The number is divisible by 9. By deleting digits whose sum is 9, we know that $a+b$ must be divisible by 9. So $a+b=0$, $a+b=9$, or $a+b=18$.

The number is divisible by 11. Then the alternative digit difference $a-b$ is divisible by 11. So $a-b=0$.

Note that $a+b$ and $a-b$ have the same parity. We have two cases:

1. $a+b=0$, $a-b=0$; 2. $a+b=18$, $a-b=0$.

In case 1, $a=b=0$. So $a\cdot b=0$.

In case 2, $a=b=9$. So $a\cdot b=81$.

Therefore, two possible values for $a\cdot b$ are 0 and 81.

Solution 2:

The two-digit sum is $S=\overline{3a}+\overline{b1}+68+00$.

Noting $\overline{3a}+\overline{b1}=\overline{ba}+31$ we have $S=\overline{ba}+99$.

S must be divisible by 99.

Note that $00\le \overline{ba}\le 99$. So $\overline{ba}=00$ or $\overline{ba}=99$.

That is, $a=b=0$ or $a=b=9$.

Therefore, $a\cdot b=0$ or $a\cdot b=81$.

Example 10

$\overline{mn41761993739701954543616}$ is divisible by 99 where m and n are digits. Find $m\cdot n$.

Answer: 64

Solution 1:

The number is divisible by 9. By deleting digits whose sum is 9 or a multiple of 9 we know that $m+n+2$ is divisible by 9. So $m+n=7$ or $m+n=16$.

The number is divisible by 11. Then the alternative digit difference $m-n+22$ is divisible by 11. So $m-n=0$.

Since $m-n$ and $m+n$ have the same parity, we have $m+n=16$, $m-n=0$.

Solving for m and n we obtain $m=n=8$.

Therefore, $m\cdot n=64$.

Solution 2:

The two-digit sum is

$$m+\overline{n4}+17+61+99+37+39+70+19+54+54+36+16$$
$$=506+\overline{nm},$$

where $m+\overline{n4}=4+\overline{nm}$ is used.

Note that $506 = 5 \times 99 + 11$ or $5 + 06 = 11$.

So $11 + \overline{nm}$ must be divisible by 99.

Therefore, $\overline{nm} = 88$. That is, $m = n = 8$.

So $m \cdot n = 64$.

Example 11

$\overline{x22222222y}$ yields a remainder of 1 upon division by 99 where *x* and *y* are digits. Find $x \cdot y$.

Answer: 2

Solution:

The two-digit sum is

$$\overline{x2} + 22 + 22 + 22 + \overline{2y} = 88 + \overline{xy}$$

where $\overline{x2} + \overline{2y} = 22 + \overline{xy}$ is used.

For $88 + \overline{xy}$ to yield a remainder of 1, $88 + \overline{xy}$ must be 100. So $\overline{xy} = 100 - 88 = 12$. That is, $x = 1$ and $y = 2$.

Therefore, $x \cdot y = 2$.

Divisibility by 999

If we similarly define the *three-digit sum* and the *final three-digit sum* for a number, we have the following theorems for divisibility by 999.

Theorem Five

A number is divisible by 999 if and only if the three-digit sum of the number is divisible by 999.

The proof is similar to the proof of theorem one.

Theorem Six

A number is divisible by 999 if and only if the final three-digit sum of the number is 999.

Example 12

Is 9876543210 divisible by 999?

Answer: No

Solution:

The three-digit sum of 9876543210 is

$$9 + 876 + 543 + 210 = 1638.$$

The final three-digit sum is $1 + 638 = 639$.

So 9876543210 is not divisible by 999.

For the remainder upon division by 999, we have the following theorem.

Theorem Seven

The final three-digit sum of a number is the remainder when the number is divided by 999.

Example 13

What is the remainder when 10203040506070809 is divided by 999?

Answer: 639

Solution:

The three-digit sum of 10203040506070809 is

$$10 + 203 + 040 + 506 + 070 + 809 = 1638.$$

The final three-digit sum is $1 + 638 = 639$.

639 is the remainder.

Example 14

$\overline{x1111y1111z}$ yields a remainder of 111 upon division by 999 where *x*, *y*, and *z* are digits. Find $x + y + z$.

Answer: 22

Solution:

The three digit sum is

$$\overline{x1} + 111 + \overline{y11} + \overline{11z} = 233 + \overline{yxz}.$$

Then $233 + \overline{yxz}$ yields remainder 111 upon division by 999. So $233 + \overline{yxz} = 999 + 111$.

We have $\overline{yxz} = 877$. That is, $x = 7$, $y = 8$, and $z = 7$.

Therefore, $x + y + z = 22$.

Divisibility by 37 and 111

Since $999 = 9 \times 111 = 27 \times 37$, the above divisibility testing method for 999 applies for the divisibility testing by 37 or by 111.

Theorem Eight

A number is divisible by 37 or 111 if and only if the three-digit sum of the number is divisible by 37 or 111.

Theorem Nine

A number is divisible by 37 or 111 if and only if the final three-digit sum of the number is divisible by 37 or 111.

Example 15

Is 77947221827 divisible by 37?

Answer: No

Solution:

The three-digit sum of 77947221827 is

$$77 + 947 + 221 + 827 = 2072.$$

The final three-digit sum is $2 + 072 = 74$.

Since 74 is divisible by 37, 77947221827 is divisible by 37.

Example 16

Is 1234321 divisible by 111?

Answer: No

Solution:

The three-digit sum of 1234321 is $1 + 234 + 321 = 556$.

556 is not divisible by 111. So 1234321 is not divisible by 111.

Example 17

What is the remainder when 112211 is divided by 37?

Answer: 27

Solution:
The three-digit sum of 112211 is

$$112 + 211 = 323 .$$

323 yields a remainder of 27 upon division by 37. So 27 is the remainder when 112211 is divided by 37.

Practice Problems

1. Circle those numbers divisible by 99.
   13579 97713 223344 112233 444555
   25578531 86420 970101 101010101
2. Find the remainder for each upon division by 99.
   123456 556677 8642 3030303 2478
   1121231234 12321 496 10101
3. $20! = \overline{ab32902008176640000}$ where a and b are digits. Find the product of a and b.
4. If six-digit number $\overline{m2468n}$ is divisible by both 9 and 11 where m and n are digits.. Find m and n.
5. If $\overline{a1b2c3456}$ is divisible by 999 where a, b, and c are digits, find $a \cdot b \cdot c$.
6. Circle those numbers divisible by 37.
   3774 123321 345456 1001 1221 2009
7. Prove that nine-digit number $\overline{abc,bca,cab}$ is always divisible by 37.
8. $37! = \overline{275275061824526abc92631959163161804800000000}$.
   Find the three digit number $\overline{abc}$.

A Problem from a Real Math Competition

Today's problem comes from MathCounts.
(MathCounts 1995 State Sprint Problem 14)
Find the sum of the x-coordinates of all possible positive integral solutions to

$$\frac{1}{x} + \frac{1}{y} = \frac{1}{7} .$$

Answer: 78
Solution:
Rewrite the equation:

$$xy - 7x - 7y = 0 .$$

Adding 49 in both sides we have

$$(x-7)(y-7) = 7^2 .$$

So we have three cases:

$$\begin{matrix} x-7=1 \\ y-7=49 \end{matrix}, \quad \begin{matrix} x-7=7 \\ y-7=7 \end{matrix}, \quad \begin{matrix} x-7=49 \\ y-7=1 \end{matrix} .$$

We obtain $x = 8$, $x = 14$, and $x = 56$ respectively.
Therefore, the answer is $8 + 14 + 56 = 78$.

Practice Problem

Find all integral solutions to

$$\frac{1}{x} + \frac{1}{y} = \frac{1}{8}$$

where $x \le y$.

Answers to All Practice Problems in Last Issue

Mental Calculation

Practice Problems I

2889	64746	986013
747	9207	8633358
37026	46953	2579742
1234436544	97433469	244332
998001	9801	99980001

Practice Problems II

2	0	1

Perimeter of a Polygon with Perpendicular Sides

Practice Problems I

1. 240	2. 72	3. 140
4. 260	5. 28	6. 40

Practice Problems II

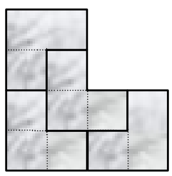
1. C	2. E	3. C
4. 2.8	5. 98	6. 20
7. 80		

A Problem from a Real Math Competition

The number is equal to
333,333,333,333 × 333,333,333,334.

Solutions to Creative Thinking Problems 61 to 63

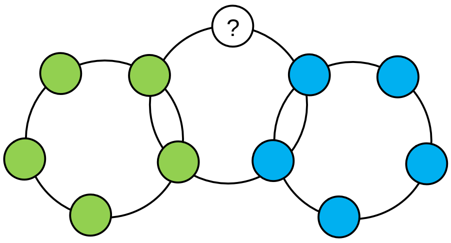
61. Four Trominos to One



62. Magic Circles

Let M be the magic number (the same sum). $M = 24$.

Both the sum of the numbers in the 5 green circles and that in the 5 blue circles are M. So the sum of the numbers in the 10 colored circles is $2M = 48$.



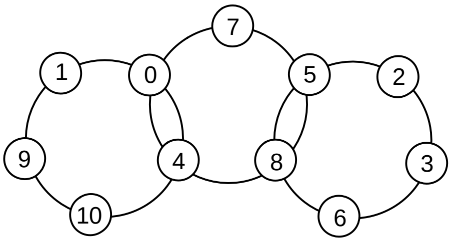
Let x be the number at the circle with the question mark.

Then the sum of all 11 numbers is $48 + x$.

On the other hand, the sum of the 11 numbers is $0+1+2+\cdots+10=55$.

Thus $48 + x = 55$. So $x = 7$. That is, the question mark must be replaced by 7.

The following is a possible filling.



63. Two Smart Students of Dr. Math

Continue on the analysis in the clue.

If the sum is 6, Dr. Math might have picked 1 and 5. Then the product could be 5. In this case Bob could figure out the two numbers because only $1\times 5 = 5$.

If the sum is 7, Dr. Math might have picked 2 and 5. Then the product could be 10. So Bob could also figure out the two numbers because only $2\times 5 = 10$.

If the sum is 8, Dr. Math might have picked 1 and 7. Then the product could be 7. In this case Bob could know $m = 1$ and $n = 7$.

If the sum is 9, Dr. Math might have picked 4 and 5. Then the product could be 20. Then Bob could also know $m = 4$ and $n = 5$.

If the sum is one of the numbers from 10 to 18, Either Al or Bob can figure out the two numbers picked by Dr. Math. You may go through all the cases by yourself.

However, if the sum is 5, Dr. Math might have picked 1 and 4, or 2 and 3. In the former case, the product passed to Bob is 4. Then Bob cannot determine the two numbers because $1\times 4 = 4$ and $2\times 2 = 4$. In the later case, the product passed to Bob is 6. Then Bob cannot determine the two numbers either because $1\times 6 = 6$ and $2\times 3 = 6$.

Therefore, if the number passed to Al is 5, Al cannot know the two numbers, and can be sure Bob does not know the two numbers either.

The sum Dr. Math tells Al is 5.

Clues to Creative Thinking Problems 64 to 66

64. One Million Words on One Square Inch

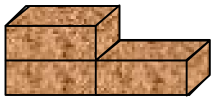
Creative Thinking Problem 52 may give you a clue.

65. Greatest 11-Digit Number

The leading digits should be as great as possible.

66. Diagonal of a Brick

This clue may give out the whole solution:



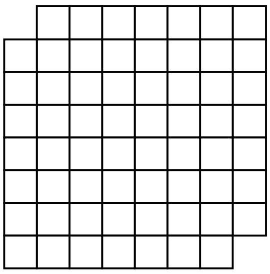
Creative Thinking Problems 67 to 69

67. Half of Eleven Is Six

I take a half from 11, but level 6. How come?

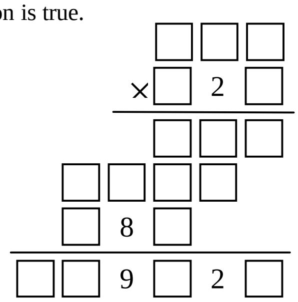
68. Domino Covering

You can use 32 dominos to cover a standard 8×8 chessboard where one domino covers exactly two squares on the board. Now two opposite corner squares are removed as shown below. Can you use 21 dominos to cover that?



69. Decoding: Another Multiplication

Fill the squares with digits such that the following multiplication is true.



(Clues and solutions will be given in the next issues.)

