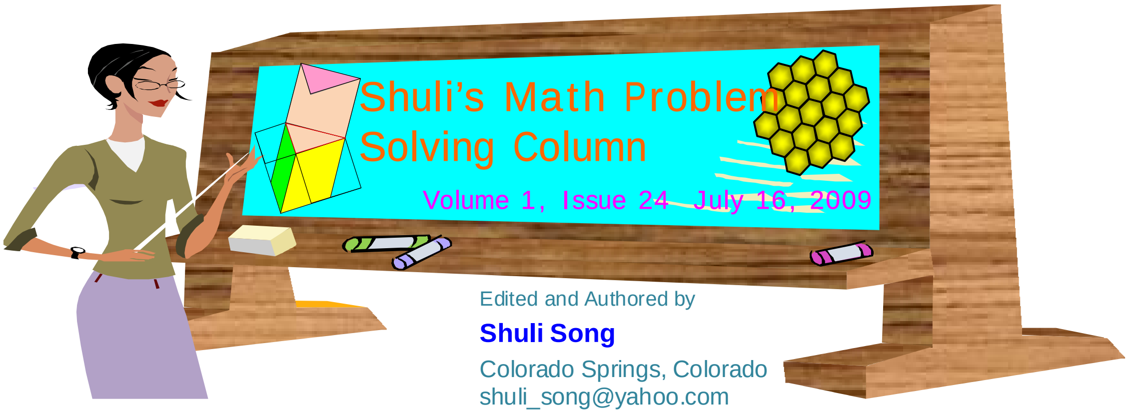


Contents

Math Trick

Mental Calculation: $\overline{a8} \times \overline{b8}$

The Trick

Mentally calculate:

$78 \times 88 =$ $\quad$ $68 \times 38 =$ $\quad$ $58 \times 28 =$

$98 \times 48 =$ $\quad$ $18 \times 28 =$ $\quad$ $88 \times 58 =$

To calculate the product of two numbers ending in 8, we have a short cut.

Write the multiplications in the general form: $\overline{a8} \times \overline{b8}$ where a and b are digits.

The steps are shown through the following examples.

Example 1

Calculate 38×68.

Step 1: Calculate $(a+1)(b+1)$.

In this example, $(a+1)(b+1) = 4 \times 7 = 28$.

Step 2: Calculate $(a+1)+(b+1)$.

In this example, $(a+1)+(b+1) = 4 + 7 = 11$.

Step 3: Multiply the result in Step 2 by 2.

In this example, $11 \times 2 = 22$

Step 4: "Subtract" the result in Step 3 from that in Step 1:

$$\begin{array}{rccc} & 2 & 8 & \\ - & & 2 & 2 \\ \hline & 2 & 5 & 8 \end{array}$$

Step 5: Attach 4 to the right of the result in step 4.

In this example, attach 4 to the right of 258: 2584.

Now we are done: $38 \times 68 = 2584$.

Example 2

Calculate 78×58.

Step 1: Calculate $8 \times 6 = 48$.

Step 2: Calculate $8 + 6 = 14$.

Step 3: Calculate $14 \times 2 = 28$.

Step 4: "Subtract":

$$\begin{array}{rccc} & 4 & 8 & \\ - & & 2 & 8 \\ \hline & 4 & 5 & 2 \end{array}$$

Step 5: Attach 4 to the right of 452: 4524.

We have $78 \times 58 = 4524$.

Why Does This Work?

$$\begin{aligned} \overline{a8} \times \overline{b8} &= [10(a+1)-2] \times [10(b+1)-2] \\ &= 100(a+1)(b+1) - 2 \cdot 10(a+1) - 2 \cdot 10(b+1) + 4 \\ &= 10\{\,10(a+1)(b+1) - 2 \cdot [(a+1)+(b+1)]\,\} + 4 \end{aligned}$$

The expression suggests the procedure shown in the above examples.

Practice Problems

$68 \times 78 =$	$18 \times 88 =$	$38 \times 58 =$
$68 \times 18 =$	$58 \times 68 =$	$98 \times 28 =$
$28 \times 48 =$	$48 \times 78 =$	$78 \times 28 =$
$88 \times 38 =$	$38 \times 98 =$	$78 \times 58 =$
$78^2 =$	$58^2 =$	$68^2 =$

Math Competition Skill

1001

A Card Magic

To perform the magic you, the magician, do the following:

1. "Randomly" take three cards from a deck of cards, and hide them.

   After this, you may blindfold yourself.
2. Ask one of your audience (or all audience together) to
1. Write any three-digit number.
2. Repeat the three-digit number to make a six-digit number. For example, if 123 is the three-digit number, the six-digit number will be 123123.
3. Randomly take a card from the three hidden cards, and divide the six-digit number by the number on the card. Throw away the remainder if there is one. Recognize A as 1, J as 11, Q as 12, and K as 13.
4. Randomly take another card from the remaining two hidden cards, and divide the quotient in step 3 by the number on the card throwing away the remainder.
5. Divide the quotient in step 4 by the original three-digit number throwing away the remainder.
6. Reveal the third card hidden.

Then the "magic" happens. The final quotient is the same as the number on the third hidden card.

How Does It Work?

Point One

Write the six-digit number by repeating a three-digit number $\overline{abc}$: $\overline{abcabc}$.

In mathematics this is equivalent to do multiplication: $\overline{abc} \times 1001$. That is, $\overline{abc} \times 1001 = \overline{abcabc}$.

Point Two

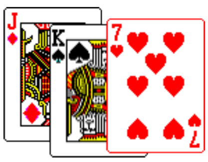
$$1001 = 7 \times 11 \times 13 .$$

That is, 1001 is divisible by 7, 11, and 13.

So a number in the form $\overline{abcabc}$ is always divisible by 7, 11, and 13.

Point Three

The three cards hidden are not randomly picked up. They are 7, J, and K, which you hid in the deck somewhere.



Point Four

It is no wonder that there is no remainder in each step of divisions.

After the first two divisions are done, the quotient is equal to $\overline{abc}$ times the number on the third card.

Now this quotient is divided by $\overline{abc}$. Of course the result is the number on the third card.

Divisibility by 7, 11, and 13

How do we test whether a number with more than 3 digits, say an 8-digit number $\overline{abcdefgh}$, is divisible by 7, 11, or 13?

We divide the digits of the umber into groups from the right, each of which has three digits:

Third Group First Group

$$\overline{ab} \quad \overline{cde} \quad \overline{fgh}$$

Second Group

The leftmost group may have fewer than 3 digits. Each group is considered as a three-digit number.

Then calculate: the number in the first group – the number in the second group + the number in the third group: $\overline{fgh} - \overline{cde} + \overline{ab}$.

Theorem One:

$\overline{abcdefgh}$ is divisible by 1001 if and only if $\overline{fgh} - \overline{cde} + \overline{ab}$ is divisible by 1001.

Since 1001 is divisible by 7, 11, and 13, we have

Theorem Two:

$\overline{abcdefgh}$ is divisible by 7, 11, or 13 if and only if $\overline{fgh} - \overline{cde} + \overline{ab}$ is divisible by 7, 11, or 13 respectively.

If there are more groups, just calculate by alternatively placing + and –.

Example 1:

Is 552692 divisible by 7?

Answer: Yes

Solution:

$692 - 552 = 140$ is divisible by 7. Then 552692 is divisible by 7.

Example 2:

Is 12839502 divisible by 13?

Answer: Yes

Solution:

$502 - 839 + 12 = -325$ is divisible by 13. Then the original number is divisible by 13.

Example 3:

Is 222333 divisible by 11?

Answer: No

Solution:

$333-222=111$ is not divisible by 11. So 222333 is not.

Proof of the Theorems

Still consider the 8 digit number $\overline{abcdefgh}$.

$$\overline{abcdefgh} = \overline{ab}\times 1{,}000{,}000 + \overline{cde}\times 1000 + \overline{fgh}$$
$$= \overline{ab}\times 999{,}999 + \overline{cde}\times 1001 + \overline{fgh} - \overline{cde} + \overline{ab}.$$

Note that $\overline{ab}\times 999{,}999$ and $\overline{cde}\times 1001$ are always divisible by 1001. Therefore, $\overline{abcdefgh}$ is divisible by 1001 if and only if $\overline{fgh}-\overline{cde}+\overline{ab}$ is divisible by 1001. This proves theorem one. Theorem two readily follows.

Math Problem Solving

Example 4:

If 9-digit number $\overline{a123b456c}$ is divisible by 7, 11, and 13 where a, b, and c are digits, find $a+b+c$.

Answer: 18

Solution:

Calculate

$$\overline{a12}-\overline{3b4}+\overline{56c} = \overline{a00}+12-304-\overline{b0}+560+c$$
$$= 268+\overline{a0c}-\overline{b0}.$$

Let $d=10-b$. Then $100-\overline{b0}=\overline{d0}$. So

$$\overline{a12}-\overline{3b4}+\overline{56c} = 168+\overline{a0c}+\overline{d0} = 168+\overline{adc}.$$

The original number is divisible by 1001. Then $168+\overline{adc}$ is divisible by 1001. Therefore, three-digit number $\overline{adc}=1001-168=833$. That is, $a=8$, $d=3$, and $c=3$. Thus $b=7$. Hence $a+b+c=18$.

Example 5:

Find the leftmost three-digits $\overline{abc}$ of

$$25! = \overline{abc11210043330985984000000}.$$

Answer: 155

Solution:

25! is divisible by 7, 11, and 13. So it is divisible by 1001. Calculate

$$S = \overline{ab}-\overline{c11}+210-043+330-985+984-000+000$$
$$= \overline{ab}-\overline{c00}+485.$$

Let $d=10-c$. Then $1000-\overline{c00}=\overline{d00}$

So $S=\overline{ab}+\overline{d00}-515=\overline{dab}-515$ is divisible by 1001. Therefore, three-digit number $\overline{dab}=515$. That is, $d=5$, $a=1$, and $b=5$. Thus $c=5$. Hence $\overline{abc}=155$.

Practice Problems

1. Circle the numbers divisible by 7.

123456789	31975251	1360194
1241661421	128395059	987654321

2. Circle the numbers divisible by 11.

123456789	31975251	1360194
1241661421	128395059	987654321

3. Circle the numbers divisible by 13.

123456789	31975251	1360194
1241661421	128395059	987654321

4. Find the leftmost three-digits $\overline{abc}$ if 12-digit number $\overline{abc123456789}$ is divisible by 1001.
5. Find $a\cdot b\cdot c$ if 12-digit number $\overline{987a654b321c}$ is divisible by 7, 11, and 13 where a, b, and c are digits.
6. $28! = \overline{30a8883b4611c13860501504000000}$ where a, b, and c are digits. Find $a\cdot b\cdot c$.

A Problem from a Real Math Competition

Today's problem comes from MathCounts.

(MathCounts 2001 State Sprint Problem 15)

A soccer ball is constructed using 32 regular polygons with equal side lengths. Twelve of the polygons are pentagons, and the rest are hexagons. A seam is sewn wherever two edges meet. What is the number of seams in the soccer ball?

Answer: 90

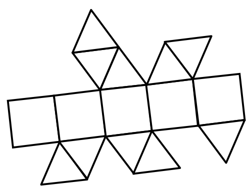
Solution:

There are 12 pentagons and $32-12=20$ hexagons. A pentagon has 5 sides. Then all 12 pentagons have $12\times 5=60$ sides. A hexagon has 6 sides. So all 20 hexagons have $20\times 6=120$ sides. Altogether there are $60+120=180$ sides in all polygons. Two sides meet to form one seam. Therefore, the number of seams is $180\div 2=90$.

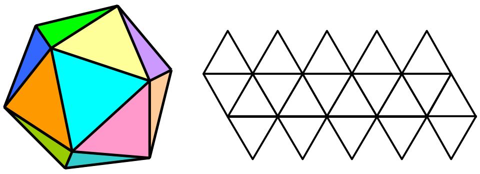
Practice Problems

1. (MathCounts 2004 State Team Problem 5)

   This net with 5 square faces and 10 equilateral triangular faces is folded into a 15-faced polyhedron. How many edges does the polyhedron have?



2. An icosahedron is a 20-faced polyhedron. The following figures show a regular icosahedron with its display.



How many edges are there in the icosahedron?

Answers to All Practice Problems in Last Issue

Mental Calculation

5451	1691	2301
1311	4071	2871
1421	4361	2291
3471	3861	4661
6241	4761	2401

Divisibility by 99 and 999

1. The numbers divisible by 99 are
   97713 223344 25578531 970101
2. 3 0 29 12 3 2 45 1 3
3. $a=2$ and $b=4$. So $a \cdot b = 8$.
4. $m=7$ and $n=0$
5. $a=3$, $b=0$, and $c=3$. So $a \cdot b \cdot c = 0$.
6. The numbers divisible by 37 are
   3774 123321 1221
7. The three-digit sum is $111 \times (a+b+c)$, which is divisible by 111, and hence by 37.
8. $\overline{abc} = 900$.

A Problem from a Real Math Competition

$(-56, 7)$, $(-24, 6)$, $(-8, 4)$, $(9, 72)$, $(10, 40)$, $(12, 24)$, $(16, 16)$

Solutions to Creative Thinking Problems 64 to 66

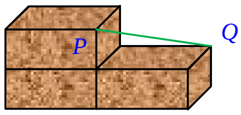
64. One Million Words on One Square Inch

65. Greatest 11-Digit Number

The answer is 95617181920.

66. Diagonal of a Brick

Measure segment *PQ*.



Clues to Creative Thinking Problems 67 to 69

67. Half of Eleven Is Six

Creative Thinking Problem 49 in *Issue* 17, *Volume* 1 may give you a clue.

67. Domino Covering

The standard chessboard coloring works.

69. Decoding: Another Multiplication

Pay attention to that 2 times a three-digit number is a four-digit number.

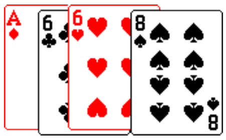
Creative Thinking Problems 70 to 72

70. $2+3 \neq 5$

When is $2+3$ not equal to 5?

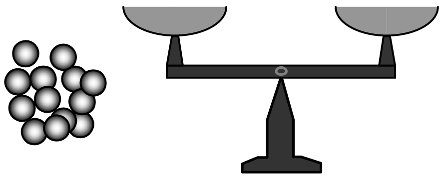
71. Another Challenge to Make 24

Make 24 with the following cards. See the rules in *Creative Thinking Problem* 6 in *Issue 2*, *Volume 1*.



72. 13 Balls

There are 13 balls that look exactly the same. One out of the 13 balls is bad. All good balls have the same weight, but the bad ball has a slightly different weight. Identify the bad ball by three times of weighing with a pan scale.



(Clues and solutions will be given in the next issues.)

