AMC/AIME Prep Club Test Solutions
August 2022

1. Calvin was asked to evaluate $37 + 31 \times a$ for some number a. Unfortunately, his paper was tilted 45 degrees, so he mistook multiplication for addition (and vice versa) and evaluated $37 \times 31 + a$ instead. Fortunately, Calvin still arrived at the correct answer while still following the order of operations. For what value of a could this have happened?

   Solution. This becomes

   $$37 + 31a = 37 \times 31 + a \quad \text{or} \quad 37 \times 30 = 30a,$$

   so $a = \boxed{37}$.

2. How many 4-digit numbers have only odd digits?

   Solution. Each digit can be 1, 3, 5, 7, or 9. This gives $5^4 = \boxed{625}$ 4-digit numbers with only odd digits.

3. The side length of a cube is increased by 100%. What is the increase in the volume of the cube, as a percentage?

   Solution. Let s be the side length of the cube beforehand, so the side length of the new cube is $2s$. Then the volume increases from s^3 to $8s^3$; this is a $\boxed{700\%}$ increase.

4. A certain two-digit number is equal to twice the sum of its digits. What is the product of its digits?

   Solution. Let $\overline{AB}$ denote the two-digit number in question. Then according to the condition,

   $$10A + B = 2(A + B) \quad \text{or} \quad 8A = B.$$

   Since $A \neq 0$ and $B \leq 9$, the only integer solution is $(A, B) = (1, 8)$, and so the requested answer is $AB = \boxed{8}$.

5. In her last game, Mary bowled 199, raising her average from 177 to 178. To raise her average to 179, what must she bowl in her next game?

   Solution. Let n denote the number of games Mary bowled when her average was 177 and let N be the score the problem asks for. According to the problem statement, the sum of her scores at this point was $177n$ and the sum of her scores after the next game was $178(n+1)$. This implies the equation

   $$177n + 199 = 178(n + 1) \quad \text{or} \quad n = 21.$$

   Applying the same reasoning as above to the next game yields

   $$178(n + 1) + N = 179(n + 2).$$

   This simplifies to

   $$N = n + (2 \cdot 179 - 178) = n + 180 = \boxed{201}.$$

6. The surface of a rectangular $9 \times 10 \times 11$ block is painted red, and the block is then cut into cubes with side length 1. Find the number of cubes that have exactly one red face.

   Solution. Notice that a cube has exactly one red face iff exactly one of its six faces is visible. This means that the cube must not be adjacent to an edge of the block (else it has at least two faces painted) and must not be tucked inside the block (else none of its faces are painted). It follows that the valid cubes are precisely those that are visible from the outside that are not bordering the edge of any face. To count the number of such cubes, first examine some 9×10 face. The interior of this face consists of a 7×8 rectangle of squares; this gives 56 cubes with exactly one face. Applying this logic to all six faces yields a count of

$$2(7 \times 8 + 8 \times 9 + 7 \times 9) = \boxed{382}.$$

7. Find the number of positive integers $1 \leq n \leq 1000$ such that n^n is a perfect square.

   Solution. We case on whether n is odd or even. If n is even, then n^n is always a perfect square, leading to 500 possibilities. If n is odd, then n^n is a perfect square precisely when n is a perfect square. This is because one can write

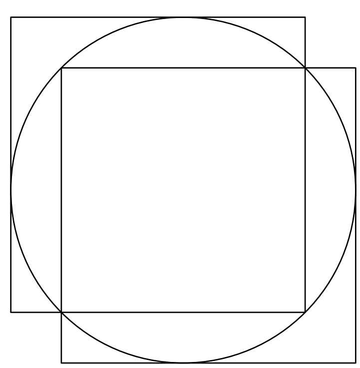
$$n^n = (n^{n-1}) \cdot n;$$

   since n^{n-1} is a perfect square by the above reasoning, we obtain the desired equivalence. The smallest and largest odd squares between 0 and 1000 are $1^2 = 1$ and $31^2 = 961$; this gives 16 new candidates. The answer is $500 + 16 = \boxed{516}$.

8. A drawer has 5 pairs of socks. Three socks are chosen at random. What is the probability that there is a pair among the three?

   Solution. Assume for now that all ten socks are distinct. There are $\binom{10}{3}$ ways to pick three socks at random. To compute the number of selections which have a pair, first observe that there are five possible pairs to choose from. After this, there are eight possible choices for the last sock. Thus there are 40 possible ways to choose a pair. The requested probability is $\frac{40}{\binom{10}{3}} = \boxed{\frac{1}{3}}$.

9. Two unit squares cover a circle of radius r, as shown below. The corresponding sides of the squares are parallel. What is r?



   Solution. Observe that the side length of the smaller square formed by the overlap of the two unit squares is $r\sqrt{2}$. Thus, the total width of the octagon is $2 - r\sqrt{2}$. This length is also equal to the diameter of the circle, or $2r$. Therefore $2 - r\sqrt{2} = 2r$, so

$$r = \frac{2}{2 + \sqrt{2}} = \boxed{2 - \sqrt{2}}.$$

10. What is the only ordered pair of real numbers (x, y) that satisfies

$$\begin{aligned} &7^x - 11y = 0 \text{ and} \\ &11^x - 7y = 0? \end{aligned}$$

Solution. Rewrite the equations as $7^x = 11y$ and $11^x = 7y$. Note that $y \neq 0$, so we may divide the second equation by the first to obtain

$$\left(\frac{7}{11}\right)^x = \frac{11}{7} = \left(\frac{7}{11}\right)^{-1},$$

upon which $x = -1$. Plugging $x = -1$ back into the system yields $y = \frac{1}{77}$, so our answer is $\boxed{(-1, \frac{1}{77})}$.

11. If p, q, and r are primes with $pqr = 7(p + q + r)$, find $p + q + r$.

Solution. Note that $7 \mid pqr$, so by the definition of prime number one of p, q, or r must be 7. WLOG let $p = 7$. Then dividing out yields

$$qr = 7 + q + r \quad \text{or} \quad (q - 1)(r - 1) = qr - q - r + 1 = 8.$$

Testing triples yields $\{q, r\} = \{3, 5\}$ as the only set of prime solutions, and so $p+q+r = 3+5+7 = \boxed{15}$.

12. Let $S = \{1, 2, 3, 4, 5, 6\}$, and consider all two-element subsets of S. What is the maximum number of these subsets we can choose so that no three, say X, Y, Z, satisfy $X \cup Y \cup Z = S$?

Solution. The answer is $\boxed{10}$. Note that this is achievable by choosing all 10 two-element subsets of $\{1, 2, 3, 4, 5\}$; then trivially no X, Y, Z satisfy $X \cup Y \cup Z = S$ since no set contains $\{6\}$. To see that 10 is the maximum, divide the two-element subsets of S into the five groups

$$\begin{aligned} &\{1, 2\}, \{3, 4\}, \{5, 6\}, \\ &\{1, 3\}, \{2, 5\}, \{4, 6\}, \\ &\{1, 4\}, \{2, 6\}, \{3, 5\}, \\ &\{1, 5\}, \{2, 4\}, \{3, 6\}, \\ &\{1, 6\}, \{2, 3\}, \{4, 5\}. \end{aligned}$$

If 11 two-element subsets were picked, by the Pigeonhole Principle three of the subsets would come from the same group; pick these three subsets to get the desired contradiction.

13. Let c be the smallest real solution to the equation

$$3^x = x + 2.$$

To six decimal places, $c = -1.87213$. Calculate the value of $3^{(3^c)}$, rounded to the nearest hundredth.

Solution. Remark that

$$3^{3^c} = 3^{c+2} = 3^c \cdot 3^2 = 9(c + 2).$$

This approximates to $9(-1.87213 + 2) \approx \boxed{1.15}$, which is the requested answer.

14. Let $ABCD$ be an isosceles trapezoid with $AD = BC$, $AB = 6$, and $CD = 10$. Suppose the distance from A to the centroid of $\triangle BCD$ is 8. Compute the area of $ABCD$.

Solution. Let G be the centroid of $\triangle BCD$, and let $X = AG \cap CD$. I claim that $BX \perp CD$. To prove this, additionally let $M = BG \cap CD$ and Y be the foot of the perpendicular from A to CD. Note that $BG : GM = 2 : 1$, so $XM = \frac{1}{2}AB$. Additionally, $YM = \frac{1}{2}AB$. Hence X and Y are symmetric about M, implying the conclusion.

To finish, remark that by similar triangles $AX = 12$, so $BX = \sqrt{12^2 - 6^2} = 6\sqrt{3}$. Hence

$$[ABCD] = \frac{1}{2}(6\sqrt{3})(6 + 10) = \boxed{48\sqrt{3}}.$$

15. How many ways are there to write the numbers from 1 to 2018 in a 2×1009 array so that each column contains exactly one even and one odd number, and no two numbers in the same row differ by 1009? Write your answer in the form $M!N!$, where M and N are positive integers. For example, one possible arrangement is shown here:

1	3	5	$\cdots$	2017
2	4	6	$\cdots$	2018

Solution. To show the generality of the solution, let $n = 1009$. For each of the pairs $(1, 1 + n), (2, 2 + n), \ldots, (n, 2n)$ exactly one number is in the top row. Since n is odd, we can choose either an even number or an odd number from each pair to be in the top row.

Let S_k be the number of legal arrangements with exactly k even numbers in the top row. There are $\binom{n}{k}$ ways to pick the even numbers in the top row, and $n!$ ways to arrange the top row. Then there are k odd numbers in the bottom row, which must go into k fixed positions, and similarly $n - k$ even numbers to go into $n - k$ fixed positions. There are $k!(n - k)!$ ways to arrange the bottom row. Multiplying all the choices and summing over k gives the total number of arrangements:

$$\sum_{k=0}^{n} \binom{n}{k} n!k!(n-k)! = \sum_{k=0}^{n} (n!)^2 = (n+1)(n!)^2 = n!(n+1)!$$

Since $n = 1009$, the answer is $\boxed{1009!1010!}$

16. For positive integer m, n, let $\gcd(m, n)$ denote the largest positive integer that is a factor of both m and n. Compute

$$\gcd(1, 91) + \gcd(2, 91) + \gcd(3, 91) + \cdots + \gcd(91, 91).$$

Solution. Note that

$$\gcd(x, 91) = \begin{cases} 91, & x = 91, \\ 13, & x = 13, 26, \ldots, 13 \cdot 6, \\ 7, & x = 7, 14, \ldots, 7 \cdot 12, \\ 1 & \text{otherwise.} \end{cases}$$

The first three cases occur 1, 6, and 12 times respectively, so the fourth case must occur exactly $91 - (1 + 6 + 12) = 72$ times. Therefore the desired sum is

$$1 \cdot 72 + 7 \cdot 12 + 6 \cdot 13 + 91 \cdot 1 = \boxed{325}.$$

OR

Solution. For all positive integers n, let

$$F(n) = \sum_{k=1}^{n} \gcd(k, n).$$

I claim that F is multiplicative. To prove this, note that if $\gcd(k, n) = d$, then $\gcd(\frac{k}{d}, \frac{n}{d}) = 1$. This implies that the number of times d is summed is $\varphi(\frac{n}{d})$, so

$$F(n) = \sum_{d|n} d\varphi\left(\frac{n}{d}\right).$$

Hence $F = \mathrm{id} * \varphi$ is the Dirichlet convolution of two multiplicative functions and is thus multiplicative.

To finish, note that if p is a prime,

$$F(p) = \sum_{k=1}^{p} \gcd(k, p) = p + 1 \cdot (p - 1) = 2p - 1.$$

Since $91 = 7 \cdot 13$, we thus have $F(91) = F(7)F(13) = 13 \cdot 25 = \boxed{325}$.