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Math Trick

Mental Calculation: $\overline{ab} \times \overline{cd}$

The Trick
Mentally calculate:

\[
\begin{align*}
32 \times 26 &= 41 \times 37 = \\
52 \times 47 &= 38 \times 24 = 
\end{align*}
\]

Write these multiplications in the general form: $ab \times cd$
where $a, b, c,$ and $d$ are digits with $a = c + 1$.

There is a short cut to do these multiplications.

Example 1
Calculate $23 \times 18$.
Recall the trick for the multiplications in the form $1a \times 1b$, which is presented in Issue 8, Volume 1.
The trick works here if we treat $23$ as $1a$ with $a = 13$.

Step 1: Calculate $1a + b$.
In this example, $23 + 8 = 31$.

Step 2: Calculate $a \times b$.
In this example, $13 \times 8 = 104$.

Step 3 “Add” them this way:

\[
\begin{array}{c}
\phantom{0}3 \\
+ 1 \\
\hline
104 \\
+ 4 \\
\hline
104 \\
\end{array}
\]

We are done: $23 \times 18 = 414$.

Example 2
Calculate $32 \times 24$.
The trick for the multiplications in the form $2a \times 2b$ works, which is presented in Issue 11, Volume 1.
Treat $32$ as $2a$ with $a = 12$.

Step 1: Calculate $2a + b$.
In this example, $32 + 4 = 36$.

Step 2: Multiply the result in step 1 by 2.
In this example, $36 \times 2 = 72$.

Step 3: Calculate $a \times b$.
In this example, $12 \times 4 = 48$.

Step 4 “Add” them:

\[
\begin{array}{c}
\phantom{0}7 \\
+ 4 \\
\hline
11 \\
+ 6 \\
\hline
176 \\
\end{array}
\]

We have $32 \times 24 = 768$.

Example 3
Calculate $51 \times 47$.
Treat $51$ as $4a$ with $a = 11$. Then this is a multiplication in the form $4a \times 4b$.

Step 1: Calculate $4a + b$.
In this example, $51 + 7 = 58$.

Step 2: Multiply the result in step 1 by 4.
In this example, $58 \times 4 = 232$. 

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Step 3: Calculate \(a \times b\).
In this example, \(11 \times 7 = 77\).

Step 4: "Add" them:

\[
\begin{array}{c}
  2 \\
  3 \\
  2 \\
  7 \\
  7 \\
\end{array}
\]

Then \(51 \times 47 = 2397\).

Example 4
Calculate \(77711\).

"Add" them:

\[
\begin{array}{c}
  2 \\
  3 \\
  2 \\
  7 \\
  7 \\
\end{array} \\
\]

Then \(23974751\).

Example 4
Calculate \(6874\).

Treat it as a multiplication in the form \(\overline{6a} \times \overline{6b}\).

Step 1: Calculate \(a \times b\).
In this example, \(74 + 8 = 82\).

Step 2: Multiply the result in step 1 by 6.
In this example, \(82 \times 6 = 492\).

Step 3: Calculate \(a \times b\).
In this example, \(14 \times 8 = 112\).

Step 4: "Add" them:

\[
\begin{array}{c}
  4 \\
  9 \\
  2 \\
  1 \\
  1 \\
  2 \\
\end{array}
\]

We get \(74 \times 68 = 5032\).

Practice Problems

\[
\begin{array}{c}
  23 \times 19 = \\
  26 \times 32 = \\
  41 \times 34 = \\
  53 \times 48 = \\
  51 \times 63 = \\
\end{array}
\]

\[
\begin{array}{c}
  16 \times 27 = \\
  36 \times 24 = \\
  38 \times 45 = \\
  54 \times 67 = \\
  73 \times 68 = \\
\end{array}
\]

Math Competition Skill

Divisibility by 9

Digit Sum and Final Digit Sum

Definitions:
The Digit Sum of a number is the sum of all digits in the number.
For example, the digit sum of 9876543210 is \(9 + 8 + 7 + 6 + 5 + 4 + 3 + 2 + 1 + 0 = 45\). The digit sum of 100,100 is \(1 + 0 + 0 + 1 + 0 + 0 + 0 = 2\).

If the digit sum has two or more digits, we calculate the digit sum for the result again until we obtain one digit. The final one digit is called the Final Digit Sum.
The digit sum of 9876543210 is 45. The digit sum of 45 is \(4 + 5 = 9\). Then 9 is the final digit sum of 9876543210.

Example 1
Find the final digit sum of 8989898989898989.
Answer: 1
Solution:
\[
\begin{array}{c}
  8 + 9 + 8 + 9 + 8 + 9 + 8 + 9 + 8 + 9 + 8 + 9 + 8 + 9 + 8 + 9 + 8 + 9 + 8 + 9 + 8 + 9 = 136 \\
  1 + 3 + 6 = 10, \\
  1 + 0 = 1. \\
\end{array}
\]

So 1 is the final digit sum.

Divisibility by 9

Let \(N\) be an integer.
How do we test whether \(N\) is divisible by 9?
We have the following theorems.

Theorem One
A number is divisible by 9 if and only if the digit sum of the number is divisible by 9.

Example 2
Is 87654321 divisible by 9?
Answer: Yes.
Solution:
Since \(8 + 7 + 6 + 5 + 4 + 3 + 2 + 1 = 36\) is divisible by 9, 87654321 is divisible by 9.

Example 3
Is 789789789789 divisible by 9?
Answer: No.
Solution:
The digit sum is 120. Since 120 is not divisible by 9, 789789789789 is not divisible by 9.

If we don't know whether 120 is divisible by 9, we can use the theorem again. Since \(1 + 2 + 0 = 3\) is not divisible by 9, 120 is not divisible by 9.

Because we can repeat using theorem one, we have the second theorem as follows.

Theorem Two
A number is divisible by 9 if and only if the final digit sum of the number is 9.

Example 4
Is 88888888888888888 divisible by 9?
Answer: No.
Solution:
The digit sum is \(21 \times 8 = 168\). The digit sum of 168 is \(1 + 6 + 8 = 15\). The digit sum of 15 is \(1 + 5 = 6\). This is the final digit sum of the original number.

So the number given is not divisible by 9.
Deleting Digits

Furthermore, we can simplify the procedure without adding digits.
We delete the digits which are 0s or 9s, or any two or more digits whose sum is 9, 18, etc. Then see what is left. If the original number is divisible by 9, 0 will be left.

Example 4
Is 726394 divisible by 9?

Answer: No.

Solution:
First we delete 9. Then we delete 7 and 2 whose sum is 9, and 6 and 3 whose sum is also 9. Then 4 is left.
Therefore, 726394 is not divisible by 9.

Example 5
Is 8841761993797019543616000000 divisible by 9?

Answer: Yes.

Solution:
First we delete 0s and 9s.
Now we delete all pairs of two digits whose sum is 9.
Then we delete all triples of three digits whose sum is 9 or 18.
Now we have nothing left.
So the original number is divisible by 9.

Proof of the Theorems

Let \( N \) be an \((n+1)\)-digit number \( a_n a_{n-1} \ldots a_1 a_0 \). Express \( N \) in the base 10 expansion:

\[
N = a_n a_{n-1} \ldots a_1 a_0 = a_n \times 10^n + a_{n-1} \times 10^{n-1} + \cdots + a_1 \times 10 + a_0
\]

\[
= a_n \times 999 \ldots 9 + a_{n-1} \times 999 \ldots 9 + \cdots + a_1 \times 9 + a_0
\]

Note that \( a_n \times 999 \ldots 9 + a_{n-1} \times 999 \ldots 9 + \cdots + a_1 \times 9 \) is always divisible by 9.

Therefore, \( N = a_n a_{n-1} \ldots a_1 a_0 \) is divisible by 9 if and only if the digit sum, is divisible by 9.

This proves theorem one.

Since we can repeat using theorem one, theorem two follows obviously.

Remainder upon Division by 9

Theorem Three
The final digit sum of a number is the remainder of the number upon division by 9.

The proof is obvious.

Example 6
What is the remainder of 86420 upon division by 9?

Answer: 2.

Solution:
The digit sum of 86420 is 20. The final digit sum is 2 + 0 = 2. Then 2 is the remainder.

We can obtain the remainder by deleting the digits as we did in the last section. If we have one digit left, this digit is the remainder. If we have nothing left, the remainder is 0. If we have several digits left, calculate the sum of the left digits. If the sum is one digit, it is the remainder. If this sum is larger than 9, calculate the digit sum of it until you obtain one digit. The one digit is the remainder.

Example 7
What is the remainder of 1234567 upon division by 9?

Answer: 1.

Solution:
2 and 7 make a 9, 3 and 6 make a 9, and 4 and 5 make another 9. Delete them. 1 is left. Then 1 is the remainder.

Example 8
What is the remainder of 2432902008176690000 upon division by 9?

Answer: 5.

Solution:
First we delete 0s and 9s.
Then we delete all pairs of two digits whose sum is 9.
Now we have the following digits left:
The sum of the left digits is 146242.
The sum of the digits in 14 is 5.

So 5 is the answer.

Examples of Problem Solving

Example 9

\[ 7m06886 \] is a 7-digit number divisible by 9. Find \( m \).

Answer: 1

Solution:
The digit sum is \( 7 + m + 0 + 6 + 8 + 8 + 6 = 35 + m \).
Note that \( m \) is a digit. That is, \( 0 \leq m \leq 9 \). The only value for \( m \) is 1 such that \( 35 + m \) is divisible by 9.

**Example 10**

\( 1234 \) is a 6-digit number yielding remainder 8 upon division by 9. Find the sum of all possible values for \( a + b \).

**Answer:** 23

**Solution:**

The digit sum is \( b + a + 2 + b + 3 + 4 = 10 + a + b \).

Note that \( a \) and \( b \) are digits. So \( 90 \leq a + b \leq 18 \).

If \( a + b = 7 \), the digit sum is 17. The remainder is the final digit sum: 8.

If \( a + b = 16 \), the digit sum is 26. The remainder is the final digit sum: 8.

No other value for \( a + b \) exists such that the property is satisfied.

Therefore, the answer is \( 7 + 16 = 23 \).

**Example 11**

\( N = 88888888 \) is a 10-digit number divisible by 72 where \( a \) and \( b \) are digits with \( 0 \leq a \leq 9 \). Find all possible values of \( a \).

**Answer:** 4, 8, and 9

**Solution:**

Note that 72 is 8 \( \times \) 9. Since 8 and 9 are relatively prime, \( N \) must be divisible by both 8 and 9.

For \( N \) to be divisible by 8, \( 8 | b \) must be divisible by 8. So \( b = 0, 4 \) or 8.

If \( b = 0 \), the digit sum of \( N \) is \( 64 + a \) For \( N \) to be divisible by 9, \( 64 + a \) must be divisible by 9. Thus \( a = 8 \).

If \( b = 4 \), the digit sum of \( N \) is \( 68 + a \). Then \( a = 4 \).

If \( b = 8 \), the digit sum of \( N \) is \( 72 + a \) We have \( a = 0 \) or 9. Since \( a > 0 \), \( a = 9 \).

Therefore, there are three possible values of \( a: 4, 8, \) and 9.

**Practice Problems**

1. Circle the numbers divisible by 9:

   \[
   234\quad 2215\quad 11609\quad 23301\quad 2229\quad 70718\quad 75161\quad 332123\quad 60606\quad 10119
   \]

2. Find the remainder for each upon division by 9:

   \[
   11111\quad 1234\quad 987\quad 1001\quad 2008\quad 12121212\quad 97531\quad 222\quad 20081105\quad 3456
   \]

3. Is \( 112200727777607680000 \) divisible by 9?

4. When \( 66044840173239439360000 \) is divided by 9, what is the remainder?

5. \( 22 \cdots 2 \) in which there are \( k \) 2’s is divisible by 9. What is the minimum value of \( k \)?

6. \( 123,4m4,3215 \) is a 9-digit number divisible by 9. Find \( m \).

7. I repeat \( 12345 \) \( m \) times to form a large number:

   \[
   123451234512345 \cdots 12345
   \]

   If the number is divisible by 9, find the possible smallest value of \( m \).

8. I wrote all natural numbers from 1 to \( n \) \((n > 10)\) together to form a large integer:

   \[
   1234567890111213 \cdots n
   \]

   If the number is divisible by 9, what is the possible smallest value of \( n \)?

9. \( N = 0000000000000 \) is an 11-digit number which is a multiple of 72, where \( a \) and \( b \) are digits with \( a > 0 \). Find \( a + b \).

**A Problem from a Real Math Competition**

Today’s problem is from University of Northern Colorado Mathematics Contest (UNCMC).

(UNCMC 2000-2001 First Round Problem 11)

35! has 41 digits when written out and can be printed as a diamond read row by row. What is the missing center digit?

**Answer:** 6

**Solution:**

35! must be divisible by 9. Why?

Thus the sum of all digits must be divisible by 9. With the following steps we can obtain the answer.

1. Add all 40 digits.
2. Find the remainder when the sum is divided by 9. Or find the final digit sum.
3. Subtract the remainder from 9. The difference is the answer.

The shortcut to get the answer is to delete digits. First delete all 0’s and 9’s. Then the table becomes
The n delete all pairs of two digits whose sum is 9. Then the table may look like:

```
1 3 3
3 1 4 7
6 6 3 8 6 1 4
4 2
1 3 3 7 5 2 3
```

Then delete all pairs of two digits whose sum is 9. Then the table may look like:

```
1
4
1
2
```

Then delete all triples of three digits whose sum is 9. Then we have:

```
1
```

Three 1’s are left, whose sum is 3. Therefore, the missing number at the center must be $9 - 3 = 6$.

**Practice Problem**

25! has 36 digits. We truncate the ending 6 zeros and write the remaining 25 digits in some order in the 5 by 5 array. What is the missing digit at the center?

```
8 8 4 1 7
1 9 5 4 6
0 1 5 1
7 6 3 4 9
9 3 7 3 9
```

**Solutions to Creative Thinking Problems 52 to 54**

52. Make One Word

My answer is “one word”.

53. Make 4 Equilateral Triangles

Use the six matchsticks to make a regular tetrahedron, which has four equilateral triangles as faces:

```
\begin{array}{cccccc}
\text{A Problem from a Real Math Competition}
\end{array}
```

54. Another Challenge to Make 24

My solution is:

```
\begin{array}{ccc}
\text{5} & \text{1} & \text{9} \\
\text{0} & \text{5} & \text{4} \\
\text{1} & \text{3} & \text{7} \\
\text{9} & \text{6} & \text{3} \\
\end{array}
```

```
\begin{array}{c}
\text{X} \\
\text{=} \\
\text{24}
\end{array}
```

**Answers to All Practice Problems in Last Issue**
**Clues to Creative Thinking Problems 55 to 57**

55. **3 x 3 Matrix**

Pay attention to the number at the center of the matrix. The sum of the nine numbers is 9 times that number.

56. **Covering with Tetrominos**

Let us color the $4 \times 5$ rectangle with the standard chessboard coloring:

![Chessboard coloring](image)

57. **Weighing Meat II**

Study from small numbers.

Again we need a weight of 1 pound for one pound of meat. For a piece of meat of 2 pounds, we may place the meat with the one-pound weight together. We can make the next weight heavier.

Now you continue.

**Creative Thinking Problems 58 to 60**

58. **A Division**

Fill the blanks with digits such that the following division expression is true.

```
  4 5 2
-----------------

  1 2 3
```

We are allowed to move the checker one square right, or up, or right-up only. That is, the checker in the figure below can be moved only to one of the squares marked by X.

![Checkerboard game](image)

Whoever moves the checker to the right-top corner is the winner.

Do you want to go first (of course to win)? Is there a winning strategy for any one?

(Clues and solutions will be given in the next issues.)