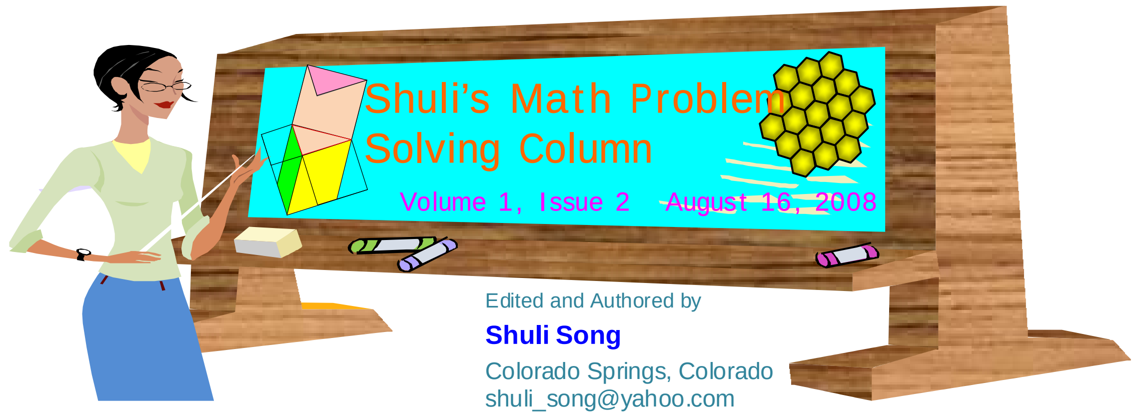

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Basic Knowledge

How Many Numbers You Count

This short lesson will answer the following questions:

How many numbers will you count if you count from 100 to 200 continuously?

How many numbers will you count if you count by 2's: 72, 74, 76, $\cdots$, up to 256?

How many numbers will you count if you count by 6's: 1, 7, 13, $\cdots$, up to 301?

The problem can be written in the general way:

How many numbers will you count if you count from m to n by k's where $\frac{n-m}{k}$ is a non-negative integer?

The answer is:

$$\frac{n-m}{k}+1.$$

There are three steps to obtain the answer:

1. Subtract the first number from the last.
2. Divide the difference by the number by which you count.
3. Add 1 to the quotient.

In the first question $m=100$, $n=200$, and $k=1$. So the answer is $\frac{200-100}{1}+1=101$.

In the second $m=72$, $n=256$, and $k=2$. The answer is $\frac{256-72}{2}+1=93$.

In the third $m=1$, $n=301$, and $k=6$. The answer is $\frac{301-1}{6}+1=51$.

Why do we add 1?

In fact, $\frac{n-m}{k}$ accounts for the number of intervals between the numbers. The number of numbers is one more than the number of intervals.

This may be understood by looking at one of your hands.

There are 5 fingers but 4 gaps between the 5 fingers.

Practice Problems

1. How many numbers will you count if you count from 123 to 456 continuously?
2. How many numbers will you count if you count by 5's: 105, 110, 115, $\cdots$, up to 300?
3. How many numbers will you count if you count by 4's: 2000, 2004, 2008, $\cdots$, up to 3000?
4. How many two-digit numbers are there?

5. How many three-digit numbers are there?
6. There are many houses on a street. The first house is numbered 3456, and the last is numbered 9876. The number difference between any two neighboring houses is 10. How many houses are there on the street?

Math Competition Skill

Model II of Balls and Sticks

Problem and Solution

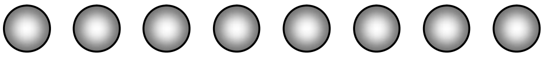
Problem 1

Five boxes are numbered 1 through 5. How many ways are there to put 8 identical balls into these boxes if empty boxes are allowed?

Answer: $\binom{12}{4} = 495$

Solution:

Arrange 8 balls in a row:

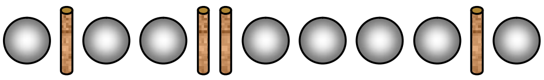


We still use four sticks to separate them. This time we may put two or more sticks together. And we may put any number of sticks at two ends.

Look at the five numbers, which are the number of balls left to the leftmost stick, the three numbers of balls between the four sticks, and the number of balls right to the rightmost stick. Any of the five numbers may be 0.

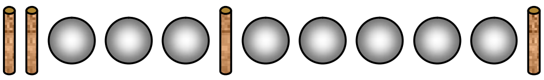
There is a one-to-one correspondence between partitions of 8 balls and distributions of 8 balls.

For example, partition



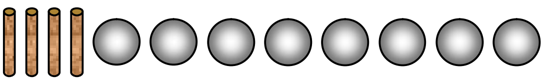
corresponds to distribution 1, 2, 0, 4, 1, which are the numbers of balls in boxes 1 to 5 respectively.

Partition

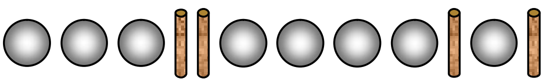


corresponds to distribution 0, 0, 3, 5, 0.

We may see more partitions:

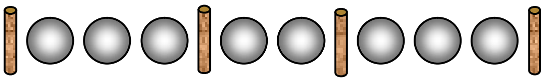


corresponding to distribution 0, 0, 0, 0, 8, and



indicating distribution 3, 0, 4, 1, 0.

Distribution 0, 3, 2, 3, 0 corresponds to partition



Now the problem becomes: how many arrangements are there to mix 4 sticks and 8 balls in a row?

In an arrangement the 12 objects (8 balls and 4 sticks) occupy 12 positions. Of the 12 positions there are 8 positions for the 8 balls, and the rest for the 4 sticks. There are $\binom{12}{8} = \binom{12}{4}$ ways to choose 8 positions from 12 for the 8 balls.

This yields the answer to the problem.

Remarks

1. In this problem empty boxes are allowed. Pay attention to this condition in a similar problem.
2. The boxes are not identical. So distribution 1, 0, 6, 1, 0 is different from distribution 0, 0, 1, 1, 6.
3. The problem can be written in the general way:

   m boxes are numbered 1 through m. How many ways are there to put n identical balls into these boxes if empty boxes are allowed?

   In this problem $n \ge m$ is not required.

   Arrange n balls and $m-1$ sticks in a row. There are $\binom{n+m-1}{n} = \binom{n+m-1}{m-1}$ ways. This is the answer.
4. In some problems we may have to figure out what are balls, and what are sticks. The following examples may give some ideas.

Similar Problems

Problem 2

How many ways are there to express 8 as the sum of 5 non-negative integers if the order of numbers in an expression is counted?

Answer: $\binom{12}{4} = 495$

Solution:

In fact, it is the same problem as Problem 1.

Arrange 4 "+" signs and 8 ones in a row.

For example, arrangement

[ ] + [ ] + [1 1 1] + [1 1 1] + [1 1]

indicates expression

$$0+0+3+3+2.$$

Expression

$$4+0+1+2+1$$

corresponds to arrangement

[1 1 1 1] + [ ] + [1] + [1 1] + [1]

There are $\binom{12}{4}$ ways to arrange 4 "+" signs and 8 ones in a row. This gives the answer to the problem.

Problem 3

A bookbinder has to bind 10 identical books using red, green, yellow, or blue covers. In how many ways can this be done?

Answer: $\binom{13}{3} = 286$

Solution:

Mix 3 sticks and 10 books in a row. There are $\binom{13}{3}$ ways to do this.

Problem 4

Thirty people vote for 5 candidates. How many possible distributions of their votes are there, if each of the 30 people votes for one candidate only? Consider only the numbers of votes to candidates. That is, all of the votes are the same.

Answer: $\binom{34}{4} = 46{,}376$

Solution:

Mix 4 sticks and 30 votes in a row. There are $\binom{34}{4}$ ways to do this.

Problem 5

There are 10 types of postcards in a post office. How many ways are there to buy 6 postcards?

Answer: $\binom{15}{6} = 5005$

Solution:

Mix 9 sticks and 6 postcards in a row. There are $\binom{15}{9} = \binom{15}{6}$ ways to do this.

Practice Problems

1. How many ways are there to express 15 as the sum of 4 non-negative integers if the order of numbers in an expression is counted?
2. A bookbinder has to bind 12 identical books using red, green, yellow, black, or blue covers. In how many ways can he do this?
3. Forty people vote for 7 candidates. How many possible distributions of their votes are there, if each of the 40 people votes for one candidate only? Consider only the numbers of votes to candidates. That is, all of the votes are the same.
4. There are 8 types of postcards in a post office. How many ways are there to buy 5 postcards?
5. In how many ways can 12 pennies be put into 5 different purses?
6. How many ways are there to cut an open necklace with 30 pearls into 8 parts if some parts may have no pearl?
7. A train with n passengers is going to make m stops. How many ways are there for passengers to get off the train at the stops if we take into account only the number of passengers who get off at each stop?
8. In how many ways can 3 people divide 6 apples, 7 oranges, and 8 pears?
9. How many ways are there to put 8 red, 8 blue, and 8 green balls into 4 different boxes?
10. How many ordered triples of non-negative integers (a, b, c) are there such that $a + b + c = 20$?

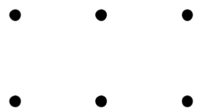
A Problem from a Real Math Competition

Today's problem comes from American Mathematics Contest Grade 8 (AMC8).

Problem

(21st AMC8 2005 Problem 21)

How many distinct triangles can be drawn using three of the dots below as vertices?



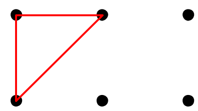
Answer: 18

Solution One:

Since the number of points is not large in this problem, we may count by systematically listing.

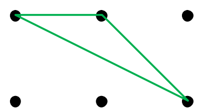
Consider all congruent triangles as one type. There are four types of triangles.

Type 1:



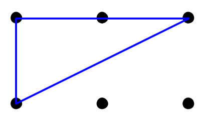
There are 8 triangles of type 1.

Type 2:



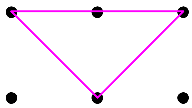
There exist 4 triangles of this type.

Type 3:



There are 4 triangles of type 3 as well.

Type 4:

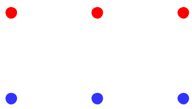


There exist 2 triangles of type 4.

Altogether, there are $8+4+4+2=18$ triangles.

Solution Two:

I color the six points below:



We observe two types of triangles.

Type 1: Two vertices are red, and one vertex is blue.

There are $\binom{3}{2}=3$ ways to choose 2 red points. For any selection of two red points we can make a triangle with any one of 3 blue points. So there are $3\times3=9$ triangles of the first type.

Type 2: Two vertices are blue, and one vertex is red.

Similarly there are 9 triangles of the second type.

Altogether, there are $9+9=18$ triangles.

Solution Three:

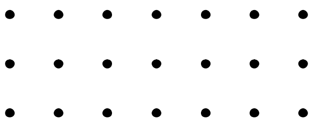
A triangle has three vertices, and a set of three points not lying in a line determines a triangle.So we will count the sets of three points. We have $\binom{6}{3}=20$ ways to select three points from six. However, there are two sets of three points lying in a line.

Therefore, there are $20-2=18$ triangles.

Practice Problem

(MathCounts 1997 National Team Problem 10)

Given a 3 by 7 rectangular array of dots, how many triangles can be formed whose vertices are dots in the array?



Answers to All Practice Problems in Last Issue

Choose *m* Elements from *n*

1. 56 5 4950 210 21 220

2. $\binom{6}{3}=20$ 3. $\binom{10}{2}=45$

4. $\binom{12}{4}=495$ 5. $\binom{9}{3}=84$

6. $\binom{50}{3}=19{,}600$

7. $\binom{50}{3}+\binom{50}{1}\cdot\binom{50}{2}=80{,}850$ 8. $\binom{12}{4}=495$

Model I of Balls and Sticks

1. $\binom{14}{3}=364$ 2. $\binom{8}{4}=70$

3. $\binom{39}{6}=3{,}262{,}623$ 4. $\binom{19}{7}=50{,}388$

5. $\binom{11}{4}=330$ 6. $\binom{29}{7}=1{,}560{,}780$

7. $\binom{n-1}{m-1}$ 8. $\binom{5}{2}\cdot\binom{7}{2}=210$

9. $\binom{7}{3}\cdot\binom{7}{3}\cdot\binom{7}{3}=42{,}875$ 10. $\binom{19}{2}=171$

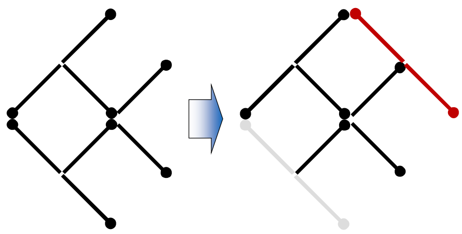
A Problem from a Real Math Competition

10

Solutions to Creative Thinking Problems 1 to 3

1. Swimming Fish

This is my solution:

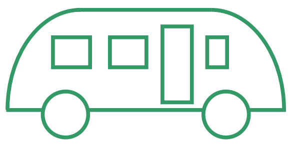


Your fish may swim in the opposite direction.

2. Moving Bus

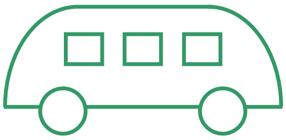
We see some windows of the bus, but we don't see the door. So the door is on the other side.

From the other side we may see



The above bus is going right.

So, when we see



the bus is going left.

More precisely, the bus is going left in the United States, but going right where people drive on the left side of roads.

3. A Careless Clockmaker

Answer:

The clock correctly shows the time 11 times at $\frac{60}{11}n$ minutes past n o'clock ($n = 1, 2, \cdots, 11$).

In detail, the 11 instances are $1:5\frac{5}{11}$, $2:10\frac{10}{11}$, $3:16\frac{4}{11}$, $4:21\frac{9}{11}$, $5:27\frac{3}{11}$, $6:32\frac{8}{11}$, $7:38\frac{2}{11}$, $8:43\frac{7}{11}$, $9:49\frac{1}{11}$, $10:54\frac{6}{11}$, and $12:00$.

Solution One:

I once gave the same problem (but only ask how many times) to my third grade sister.

She used a real traditional clock to find the answer which is 11. She knew that the clock shows the correct time when the two hands meet. She told me that after 12 o'clock about 1:05 is the first instance at which the two hands meet, about 2:10 is the second instance, etc.

From that point I tried to teach her to understand the exact instances at which the two hands meet.

This is what I did:

Suppose that her observation is correct. That is, 1:05 is the first instance. Then 2:10 is the second, 3:15 is the third etc. That is, the two hands meet once every 1 hour 5 minutes.

Furthermore, we eventually have 11:55 as the eleventh instance. It is obviously incorrect. 12:00 is the eleventh instance. We have an error of 5 minutes.

How?

We know that 1 hour 5 minutes is not accurate. The exact time interval is 1 hour and 5 + a little more minutes.

Because we have ignored the "little more", we eventually get 11:55 as the eleventh instance with an accumulated error of 5 minutes. So we have to distribute the 5 minutes equally into the 11 intervals.

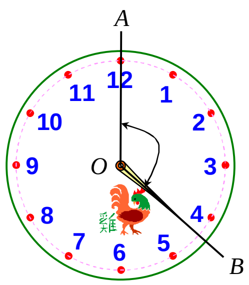
Now we know that the "little more" is $\frac{5}{11}$. So The exact time interval is 1 hour and $5\frac{5}{11}$ minutes.

The accurate first instance is $5\frac{5}{11}$ minutes past 1 o'clock, the second is $10\frac{10}{11}$ minutes past 2 o'clock, the third is $15\frac{15}{11} = 16\frac{4}{11}$ minutes past 3 o'clock, etc.

Solution Two:

Assume that the two hands meet at x minutes past n o'clock, $n = 1, 2, \cdots, 11$. Here x depends on n.

Let O be the center of the clock, OA be the ray pointing 12 o'clock, and OB be the ray pointing x minutes past n o'clock.



Remember the facts:

1. The minute hand moves $\frac{360°}{60} = 6°$ every minute.
2. The hour hand moves $\frac{360°}{12} = 30°$ every hour.

Looking at the minute hand we have in degrees

$$\angle AOB = 6x .$$

Note that x minutes accounts for $\frac{x}{60}$ hours.

Looking at the hour hand we observe in degrees

$$\angle AOB = 30\left(n + \frac{x}{60}\right).$$

So we have

$$6x = 30\left(n + \frac{x}{60}\right).$$

Solving for x we obtain

$$x = \frac{60}{11}n .$$

Thus the two hands meet at $\frac{60}{11}n$ minutes past n o'clock ($n = 1, 2, \cdots, 11$).

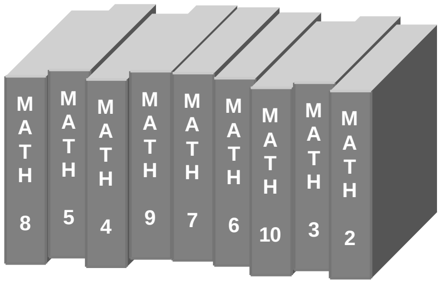
For $n = 1$, $x = \frac{60}{11} = 5\frac{5}{11}$, for $n = 2$, $x = \frac{120}{11} = 10\frac{10}{11}$, for $n = 3$, $x = \frac{180}{11} = 16\frac{4}{11}$, etc.

For $n = 11$, $x = 60$. The time instance is 12:00.

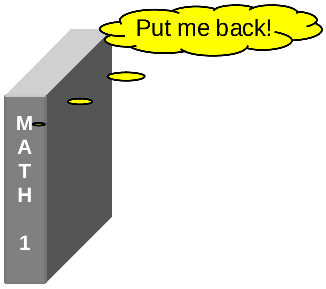
Creative Thinking Problems 4 to 6

4. Sorting Books

At a friend's party, I casually took a book from his bookshelf. It was Volume 1 of a series of 10 volumes. When I tried putting the book back later, I had trouble remembering where it was placed. I knew that my friend had a very special rule for sorting his books but had no idea what it was. The remaining nine volumes were arranged this way:



Please help me figure out where Volume 1 should be placed?



5. Four Liters of Water

Say you have a container that can hold 3 liters of water and another container that can hold 5 liters of water. Describe a process that will yield 4 liters of water if you have access to a water tap with unlimited water.

6. Make 24 with 3, 3, 7, and 7

Making 24 is one of my favorite games. To play this game, first take out all of the face cards from a standard deck, leaving forty cards with four cards of each number from 1 to 10. Randomly draw four cards. Using the four numbers on the cards, try to create 24 with operations: +, –, $\times$, and $\div$. Parentheses are allowed.

(Two or more friends can play together. The first person to make 24 correctly gets all four cards and the person with the most cards at the end wins.)

For example: if the four cards are 3, 3, 9, and 2 shown in Figure 1,

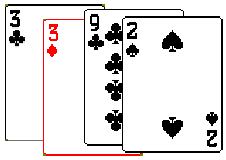


Figure 1

we can make 24 in the following ways:

$$2\times 9+3+3=24,$$

$$3\times(9-2)+3=24,$$

$$3\times(9+2-3)=24.$$

If 6, 8, 3, and 5 are drawn, shown in Figure 2,

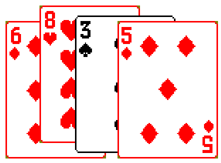


Figure 2

we can make 24 in the following ways:

$$3\times 8\times(6-5)=24,$$

$$(5-6\div 3)\times 8=24,$$

$$6\times 8\div(5-3)=24.$$

Now it's your turn.

Make 24 using the four cards shown in Figure 3.

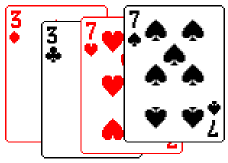


Figure 3

(Solutions will be presented in the next issue.)

