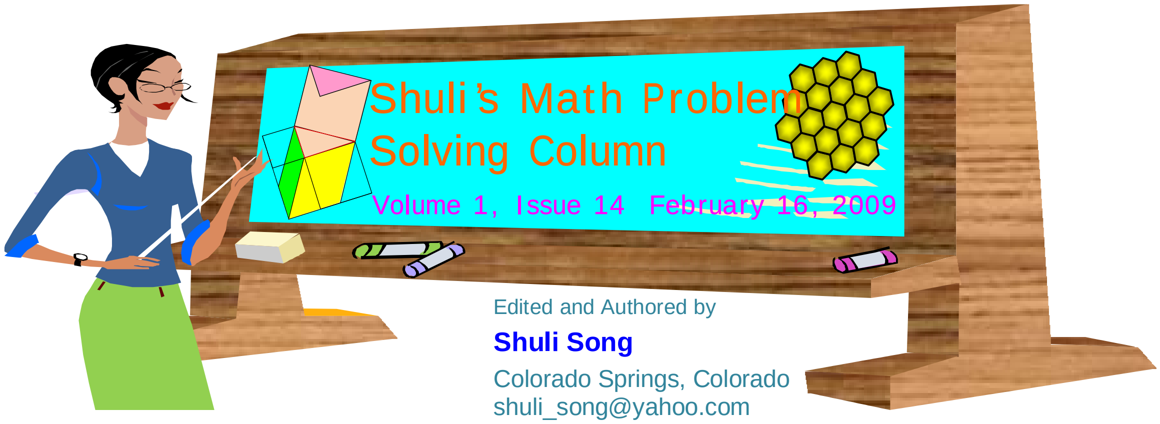

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Content

Math Trick

Mental Calculation: $\overline{ab} \times \overline{cd}$

The Trick

This short lesson will teach how to obtain the products for the following multiplications mentally.

$38 \times 42 =$ $\quad 51 \times 69 =$ $\quad 84 \times 76 =$

$68 \times 52 =$ $\quad 47 \times 53 =$ $\quad 95 \times 85 =$

There are two properties in common for all these pairs of two-digit numbers:

1. The sum of the two ones digits is 10.
2. One tens digit is one more than the other tens digit.

We write the multiplications in the general form: $\overline{ab} \times \overline{cd}$ where a, b, c, and d are digits with $b + d = 10$ and $a + 1 = c$.

Following the given steps you will be able to complete each multiplication in 3 seconds.

Example 1

Calculate 74×86.

Step 1: Calculate c^2.

In this example, $8^2 = 64$.

Step 2: Calculate d^2.

In this example, $6^2 = 36$.

Step 3: "Subtract" d^2 from c^2 this way:

$$\begin{array}{rcccc} & 6 & 4 & & \\ - & & & 3 & 6 \\ \hline & 6 & 3 & 6 & 4 \end{array}$$

Now we are done: $74 \times 86 = 6364$.

Example 2

Calculate 42×38.

Note that $\overline{cd}$ is always the larger number. In this problem $\overline{cd} = 42$ and $\overline{ab} = 38$.

Step 1: Calculate $4^2 = 16$.

Step 2: Calculate $2^2 = 4$, treated as two digits: 04

Step 3: "Subtract" 04 from 16:

$$\begin{array}{rcccc} & 1 & 6 & & \\ - & & & 0 & 4 \\ \hline & 1 & 5 & 9 & 6 \end{array}$$

We have $42 \times 38 = 1596$.

It works for numbers with three or more digits.

Example 3

Calculate 243×257.

Step 1: Calculate $25^2 = 625$.

Step 2: Calculate $7^2 = 49$.

Step 3: "Subtract" 49 from 625:

$$\begin{array}{rccccc} & 6 & 2 & 5 & & \\ - & & & & 4 & 9 \\ \hline & 6 & 2 & 4 & 5 & 1 \end{array}$$

We obtain $243 \times 257 = 62451$.

Example 4

Calculate 1204×1196.

Step 1: Calculate $120^2 = 14400$.

Step 2: Calculate $4^2 = 16$.

Step 3: "Subtract" 16 from 14400:

$$\begin{array}{rrrrrrr} & 1 & 4 & 4 & 0 & 0 & \\ - & & & & & 1 & 6 \\ \hline & 1 & 4 & 3 & 9 & 9 & 8 & 4 \end{array}$$

So $1204 \times 1196 = 1439984$.

Why Does This Work?

Write $\overline{ab}$ and $\overline{cd}$ in the base 10 representation:

$$\overline{ab} = 10a + b \text{ and } \overline{cd} = 10c + d.$$

So we have

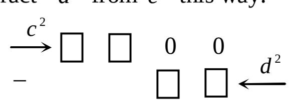
$$\overline{ab} \times \overline{cd} = (10a + b) \times (10c + d)$$
$$= (10a + 10 - d) \times (10c + d) = [10(a+1) - d] \times (10c + d)$$
$$= (10c - d)(10c + d) = (10c)^2 - d^2 = 100c^2 - d^2$$

This shows that to calculate $\overline{ab} \times \overline{cd}$, we may do

Step 1: Calculate c^2.

Step 2: Calculate d^2.

Step 3: "Subtract" d^2 from c^2 this way:



Practice Problems I

$16 \times 24 =$	$98 \times 82 =$	$69 \times 71 =$
$68 \times 72 =$	$27 \times 33 =$	$54 \times 46 =$
$75 \times 85 =$	$43 \times 37 =$	$62 \times 58 =$
$76 \times 64 =$	$35 \times 45 =$	$31 \times 29 =$
$88 \times 72 =$	$43 \times 57 =$	$94 \times 86 =$

Practice Problems II

$124 \times 116 =$	$97 \times 103 =$	$135 \times 145 =$
$397 \times 403 =$	$256 \times 244 =$	$101 \times 119 =$
$1197 \times 1203 =$	$1056 \times 1044 =$	$278 \times 262 =$
$1155 \times 1145 =$	$2109 \times 2091 =$	$3092 \times 3088 =$

Math Competition Skill

All Roads to Rome

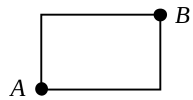
All roads lead to Rome.
– Chinese Proverb

In this lesson we will count how many roads there are to Rome.

We will use a method called "Marking Numbers" to count the number of routes.

Example 1

The figure below is a road map. How many shortest routes are there from *A* to *B*? Note that "shortest" means that routes always approach the destination. In this problem we can only go up and right.



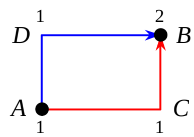
Answer: 2

Solution:

Obviously there are two routes:

(1) Red route: from *A* to *C*, and then to *B*;

(2) Blue route: from *A* to *D*, and then to *B*.



Now we place numbers. We mark 1 at the starting point.

To *C* we can go only from *A*. We bring the number at *A* to *C*, which is 1.

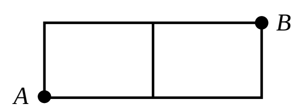
Similarly 1 is marked at *D*.

To *B* we can go from either *C* or *D*. We bring the sum of the numbers at *C* and *D* to *B*, which is $1 + 1 = 2$.

The answer to the problem is 2.

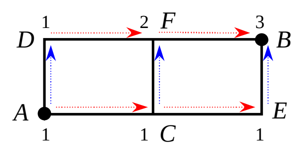
Example 2

How many shortest routes are there from *A* to *B* in the figure below?



Answer: 3

Solution:



To *B* from *E* or from *F*

To *F* from *C* or from *D*

To *E* from *C*

To *D* from *A*

To *C* from *A*

Mark numbers at the cross-sections. First mark 1 at *A*.

At *C* 1, brought from *A*.

At *D* 1, brought from *A*

At *E* 1, brought from *C*

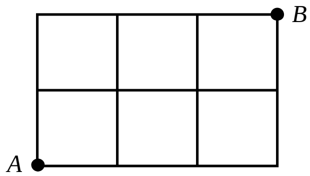
At *F* 2 = 1 + 1, brought from *C* and *D*

At *B* 3 = 1 + 2, brought from *E* and *F*

Therefore, there are 3 shortest routes, which are *ACEB*, *ACFB*, and *ADFB*.

Example 3

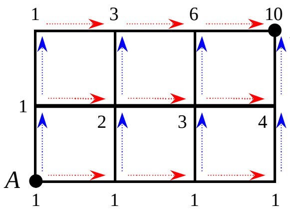
The figure shows a road map. How many shortest routes are there from A to B?



Answer: 10

Solution:

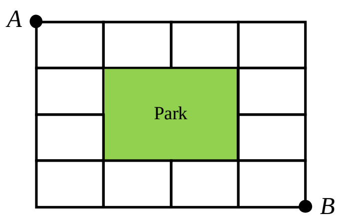
Mark numbers starting at A:



The answer is 10.

Example 4

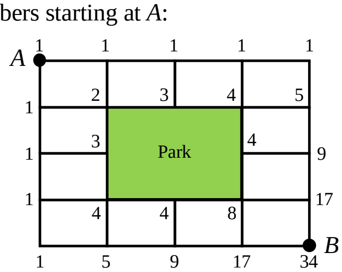
The figure shows a road map. How many shortest routes are there from A to B?



Answer: 34

Solution:

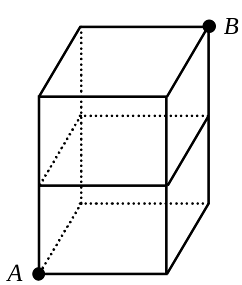
Mark numbers starting at A:



There are 34 shortest routes from A to B.

Example 5

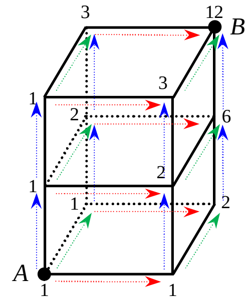
An ant wants to climb along the cucumber trellis from A to B. How many shortest routes can the ant find?



Answer: 12

Solution:

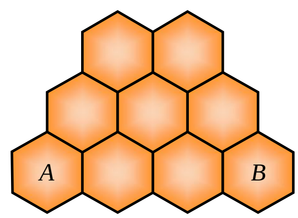
Mark numbers starting at A:



The ant can find 12 shortest routes.

Example 6

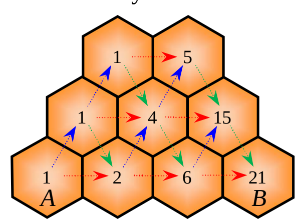
A bee wants to creep on the honeycomb from hole A to hole B. If it is allowed to creep right (up-right, down-right or directly right), how many different routes can the bee find?



Answer: 21

Solution:

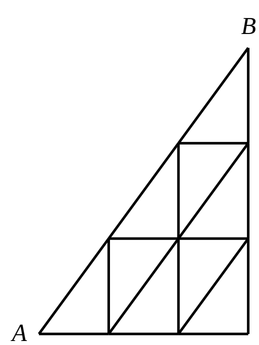
Mark numbers in the honeycomb holes:



The bee can find 21 different routes.

Example 7

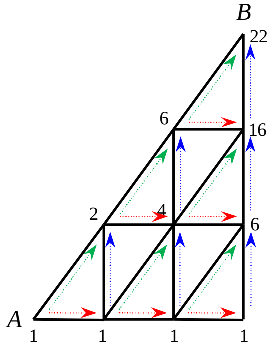
If you are allowed to move only up, right or up-right, how many different routes are there?



Answer: 22

Solution:

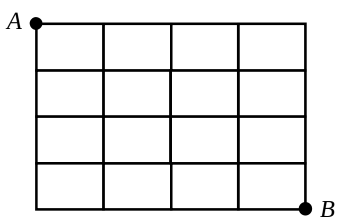
Mark the numbers as follows:



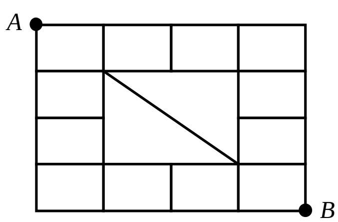
There are 22 different routes from *A* to *B*.

Practice Problems I

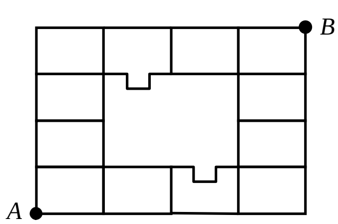
1. How many shortest routes are there from *A* to *B* in the following road map?



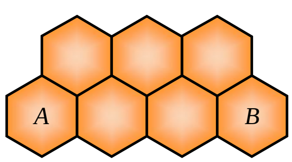
2. How many routes are there from *A* to *B* in the following road map if you are allowed to move down, right or down-right?



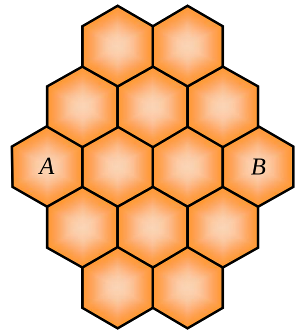
3. How many shortest routes are there from *A* to *B* in the following road map? Pay attention to the "shortest" routes.



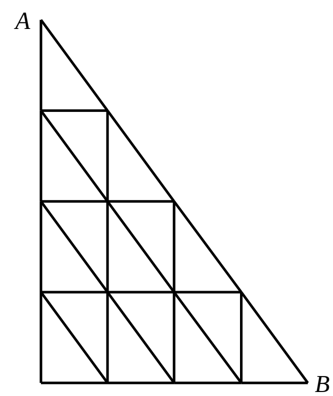
4. A bee wants to creep on the honeycomb from hole *A* to hole *B*. If it is allowed to creep right (up-right, down-right or directly right), how many different routes can the bee find?



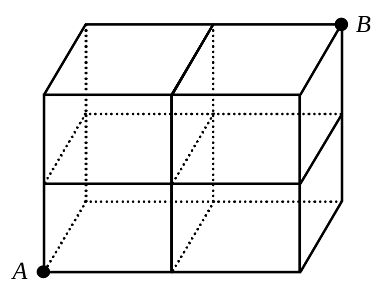
5. A bee wants to creep on the honeycomb from hole A to hole B. If it is allowed to creep right (up-right, down-right or directly right), how many different routes are there?



6. If you are allowed to move only down, right or down-right, how many different routes are there?



7. An ant wants to climb along the cucumber trellis from *A* to *B*. How many shortest routes can the ant find?

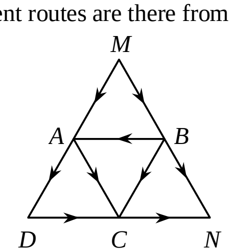


Practice Problems II

(Real Math Competition Problems)

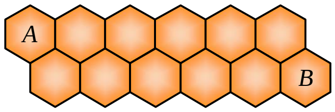
1. (2nd AMC8 1986 Problem 9)

   Using only the paths and the directions shown, how many different routes are there from *M* to *N*?



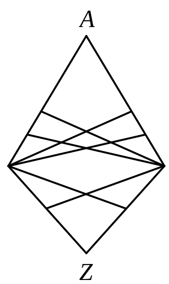
2. (MathCounts State Sprint 1996 Problem 21)

There are three allowable moves in the portion of the bee hive shown: from one cell to a cell directly to the right; from one cell to an adjacent cell which is up and to the right; or from a cell to a bordering cell which is down and to the right. How many distinct paths are there from Cell A to cell B?



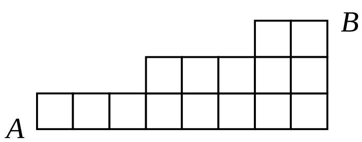
3. (MathCounts National Team 1999 Problem 1)

Using only paths that follow the line segments and go downward, how many different paths go from A to Z?



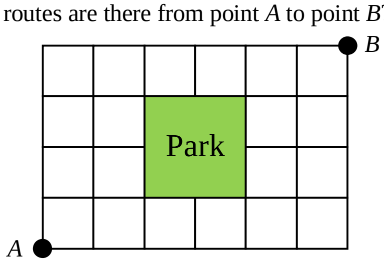
4. (MathCounts National Sprint 2000 Problem 21)

Moves are only allowed one segment to the right or one segment up. How many paths from A to B are possible?



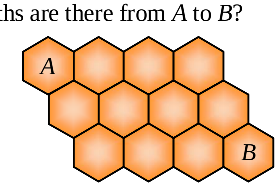
5. (1st Canadian Open Mathematics Challenge 1996 Problem 5)

A road map of Grid City is shown in the diagram. The perimeter of the part is a road but there is no road through the park. How many different shortest road routes are there from point A to point B?



6. (MathCounts Chapter Team 2002 Problem 10)

In the hexagonal grid, you may step from your current hexagon to any adjacent hexagon. How many 5-step paths are there from A to B?

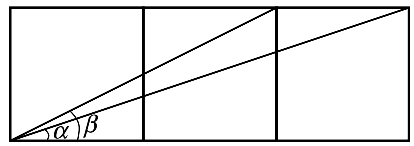


A Problem from a Real Math Competition

Today's problem comes from the British Columbia Junior High School Mathematics Contest (BCJHSMC).

(BCJHSMC 2004 Final Round Problem B5)

The diagram shows three congruent squares. Find the measure of the angle $\alpha+\beta$.

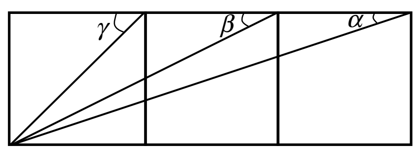


Answer: $45°$

Many math competitions ask you to prove that $\alpha+\beta=45°$.

I can also presented as follows:

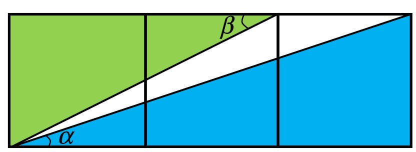
Prove that $\alpha+\beta+\gamma=90°$ in the figure.



Solution One (Trigonometry):

Look at the blue right triangle. We know $\tan\alpha=\frac{1}{3}$.

Look at the green right triangle. We know $\tan\beta=\frac{1}{2}$.

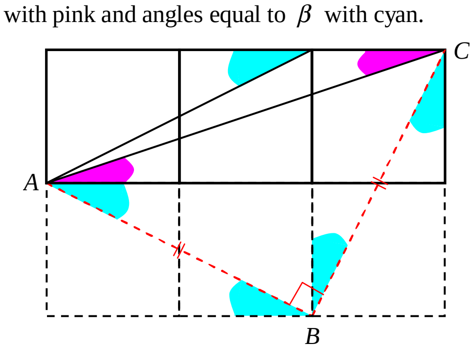


So $\tan(\alpha+\beta)=\frac{\tan\alpha+\tan\beta}{1-\tan\alpha\tan\beta}=\frac{\frac{1}{3}+\frac{1}{2}}{1-\frac{1}{3}\cdot\frac{1}{2}}=1$.

Therefore, $\alpha+\beta=45°$.

Solution Two (Geometry):

Attach three congruent squares. We mark angles equal to α with pink and angles equal to β with cyan.

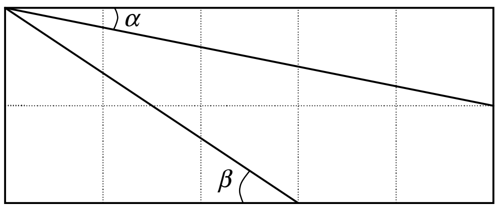


We observe that ΔABC is an isosceles right triangle with $BA = BC$.

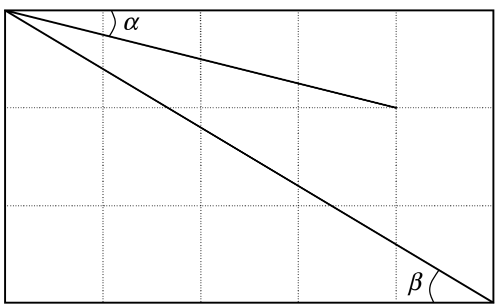
Now we see that a pink angle and a cyan angle sum to $45°$. That is, $\alpha + \beta = 45°$.

Practice Problems

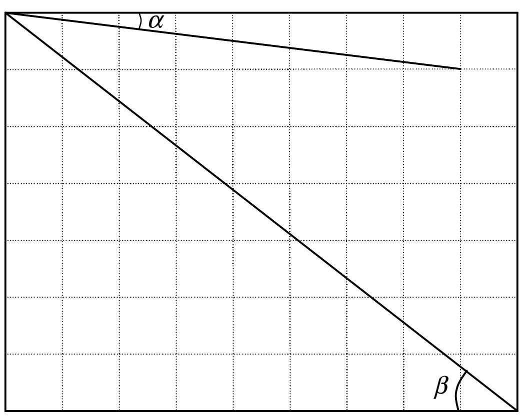
1. The diagram shows a 2 × 5 square grid. Prove that $\alpha + \beta = 45°$.



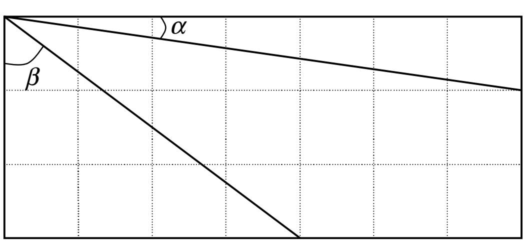
2. The diagram shows a 3 × 5 square grid. Prove that $\alpha + \beta = 45°$.



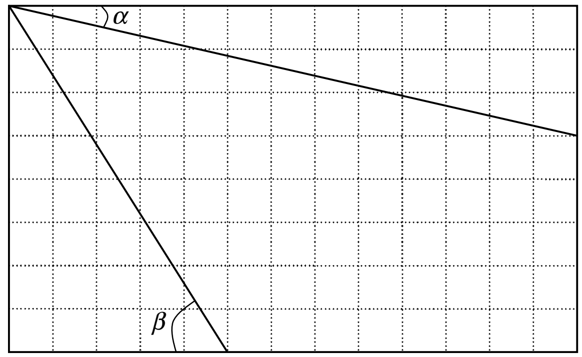
3. The diagram shows a 7 × 9 square grid. Prove that $\alpha + \beta = 45°$.



4. The diagram shows a 3 × 7 square grid. Prove that $\beta - \alpha = 45°$.



5. The diagram shows an 8 × 13 square grid. Prove that $\beta - \alpha = 45°$.



Answers to All Practice Problems in Last Issue

Math Trick: Mental Calculation

Practice Problems I

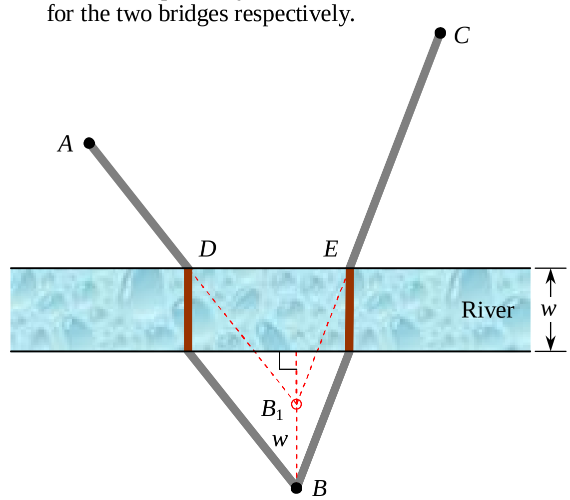
42218	42021	42432
42642	41410	43054
42009	42224	43472
41412	42230	41814
43264	42849	42436

Practice Problems II

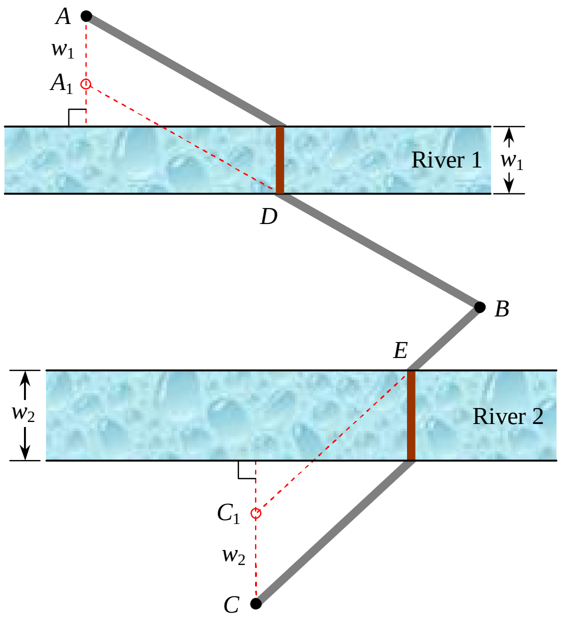
93942	95481	93940
165236	164424	166872
253512	255530	257049
369664	366024	497724
492804	647218	820836

Where to Build a Bridge?

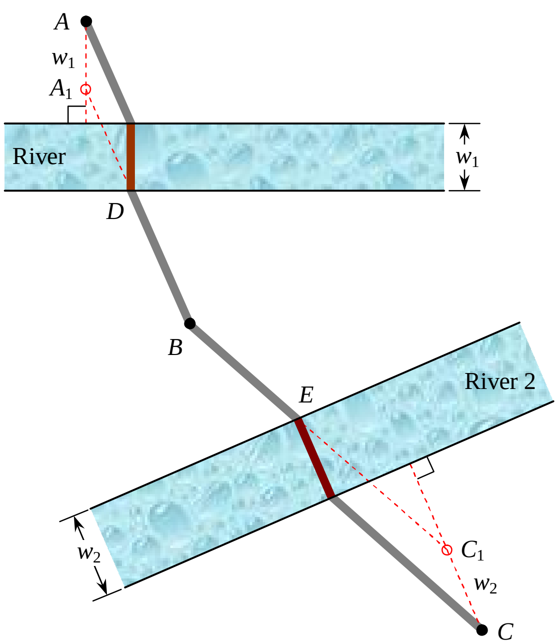
1. Let w be the width of the river. Perpendicularly to the south shore, translate B by w towards the river to B_1. Draw AB_1 and CB_1, intersecting the north shore at D and E respectively. Then D and E are the locations for the two bridges respectively.



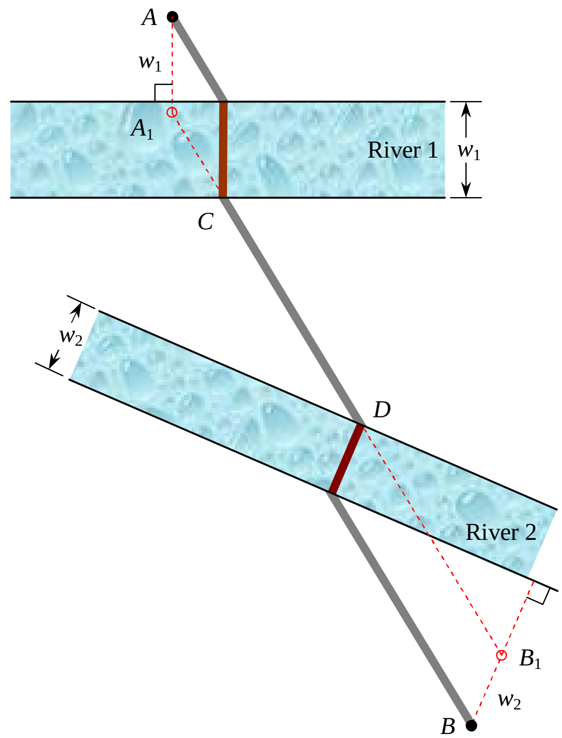
2. Let w_1 and w_2 be the widths of River 1 and River 2 respectively. Perpendicularly to the north shore of River 1, translate A by w_1 towards River 1 to A_1. Perpendicularly to the south shore of River 2, translate C by w_2 towards River 2 to C_1. Draw A_1B intersecting the south shore of River 1 at D. Draw C_1B intersecting the north shore of River 2 at E. Then D and E are the locations for the two bridges respectively.



3. As shown below, D and E are the locations for the two bridges respectively.
   The construction procedure is the same as in the last problem.



4. Let w_1 and w_2 be the widths of River 1 and River 2 respectively. Perpendicularly to the north shore of River 1, translate A by w_1 towards River 1 to A_1. Perpendicularly to the south shore of River 2, translate B by w_2 towards River 2 to B_1. Draw A_1B_1 intersecting the south shore of River 1 and the north shore of River 2 at C and D respectively. Then C and D are the locations for the two bridges respectively.



A Problem from a Real Math Competition

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Solutions to Creative Thinking Problems 34 to 36

34. Make a Tent

This is my solution:

Would you say that "this is a tent"?

35. Filling 1 to 9

Note that $4234 = 2 \times 29 \times 73$.

$2 \times 73 = 146$ is the only choice for the three digit number.

Then the number multiplied to 146 must be 29.

Now we know that the other two numbers are 58 and 73.

So the solution is:

$\boxed{5}\boxed{8} \times \boxed{7}\boxed{3} = \boxed{2}\boxed{9} \times \boxed{1}\boxed{4}\boxed{6} = 4234$

Each digit from 1 to 9 is used once and only once.

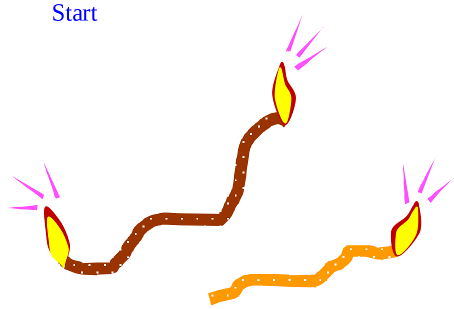
36. Two Ropes

We need 1 hour to burn a rope from one end to the other end. If we light two ends of a rope simultaneously, we need 30 minutes to burn the rope.

The solution follows.

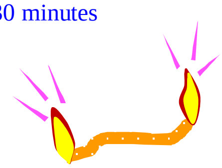
Light two ends of one rope and one end of the other rope simultaneously.

Start



When the first rope is burnt, 30 minutes has passed. At this moment light the other end of the second rope.

In 30 minutes



Then 15 minutes more is needed to burn the second rope.

We have a total of 45 minutes.

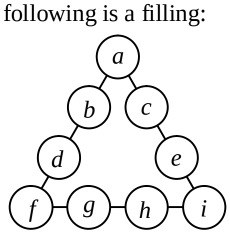
Clues to Creative Thinking Problems 37 to 39

37. True or False Again

For this kind of logic problems, assume that a statement is true one by one, and examine whether we will get any contradictions.

38. Another Magic Triangle

Assume that the following is a filling:



where *a*, *b*, *c*, *d*, *e*, *f*, *g*, *h*, and *i* are 1 to 9 in some order.

Then think of the three numbers at the vertices.

39. 10 × 10 Matrix

Study some regular fillings such as the following two and observe any results you may obtain.

+0	+1	+2	+3	+4	−5	−6	−7	−8	−9
+10	+11	+12	+13	+14	−15	−16	−17	−18	−19
+20	+21	+22	+23	+24	−25	−26	−27	−28	−29
+30	+31	+32	+33	+34	−35	−36	−37	−38	−39
+40	+41	+42	+43	+44	−45	−46	−47	−48	−49
−50	−51	−52	−53	−54	+55	+56	+57	+58	+59
−60	−61	−62	−63	−64	+65	+66	+67	+68	+69
−70	−71	−72	−73	−74	+75	+76	+77	+78	+79
−80	−81	−82	−83	−84	+85	+86	+87	+88	+89
−90	−91	−92	−93	−94	+95	+96	+97	+98	+99

+0	−1	+2	−3	+4	−5	+6	−7	+8	−9
−10	+11	−12	+13	−14	+15	−16	+17	−18	+19
+20	−21	+22	−23	+24	−25	+26	−27	+28	−29
−30	+31	−32	+33	−34	+35	−36	+37	−38	+39
+40	−41	+42	−43	+44	−45	+46	−47	+48	−49
−50	+51	−52	+53	−54	+55	−56	+57	−58	+59
+60	−61	+62	−63	+64	−65	+66	−67	+68	−69
−70	+71	−72	+73	−74	+75	−76	+77	−78	+79
+80	−81	+82	−83	+84	−85	+86	−87	+88	−89
−90	+91	−92	+93	−94	+95	−96	+97	−98	+99

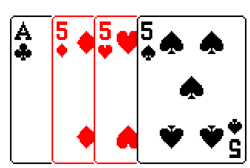
Creative Thinking Problems 40 to 42

40. The Average of 4 and 6 is not 5!

You go up a mountain with the average speed of 4 km/hr., and down the mountain on the same route with 6 km/hr. What is your average speed for the whole trip?

41. Making 24 with 1, 5, 5, and 5

Make 24 with



See the rules in Creative Thinking Problem 6 appearing in *Issue 2, Volume 1.*

42. Aging Faster?

I know a girl. The day before yesterday she was 13. Next year she will be 16. What day is it? When is her birthday?

(Clues and solutions will be given in the next issues.)