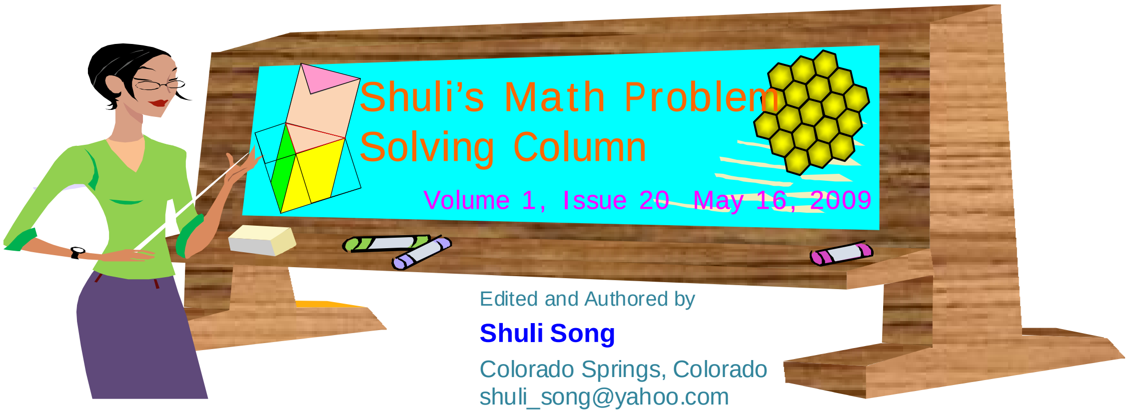

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Contents

Math Trick

Mental Calculation: $\overline{ab} \times \overline{cd}$

The Trick

Mentally calculate:

$32 \times 26 =$ $\quad 24 \times 18 =$ $\quad 41 \times 37 =$

$52 \times 47 =$ $\quad 45 \times 39 =$ $\quad 38 \times 24 =$

Write these multiplications in the general form: $\overline{ab} \times \overline{cd}$ where a, b, c, and d are digits with $a = c + 1$.

There is a short cut to do these multiplications.

Example 1

Calculate 23×18.

Recall the trick for the multiplications in the form $\overline{1a} \times \overline{1b}$, which is presented in *Issue* 8, *Volume* 1. The trick works here if we treat 23 as $\overline{1a}$ with $a = 13$.

Step 1: Calculate $\overline{1a} + b$.

In this example, $23 + 8 = 31$.

Step 2: Calculate $a \times b$.

In this example, $13 \times 8 = 104$.

Step 3 "Add" them this way:

$$\begin{array}{rccc} & 3 & 1 & \\ + & 1 & 0 & 4 \\ \hline & 4 & 1 & 4 \end{array}$$

We are done: $23 \times 18 = 414$.

Example 2

Calculate 32×24.

The trick for the multiplications in the form $\overline{2a} \times \overline{2b}$ works, which is presented in *Issue* 11, *Volume* 1. Treat 32 as $\overline{2a}$ with $a = 12$.

Step 1: Calculate $\overline{2a} + b$.

In this example, $32 + 4 = 36$.

Step 2: Multiply the result in step 1 by 2.

In this example, $36 \times 2 = 72$.

Step 3: Calculate $a \times b$.

In this example, $12 \times 4 = 48$.

Step 4 "Add" them:

$$\begin{array}{rccc} & 7 & 2 & \\ + & & 4 & 8 \\ \hline & 7 & 6 & 8 \end{array}$$

We have $32 \times 24 = 768$.

Example 3

Calculate 51×47.

Treat 51 as $\overline{4a}$ with $a = 11$. Then this is a multiplication in the form $\overline{4a} \times \overline{4b}$.

Step 1: Calculate $\overline{4a} + b$.

In this example, $51 + 7 = 58$.

Step 2: Multiply the result in step 1 by 4.

In this example, $58 \times 4 = 232$.

Step 3: Calculate $a \times b$.

In this example, $11 \times 7 = 77$.

Step 4 "Add" them:

$$\begin{array}{ccccc} & 2 & 3 & 2 & \\ + & & & 7 & 7 \\ \hline & 2 & 3 & 9 & 7 \end{array}$$

Then $51 \times 47 = 2397$.

Example 4

Calculate 74×68.

Treat it as a multiplication in the form $\overline{6a} \times \overline{6b}$.

Step 1: Calculate $\overline{6a} + b$.

In this example, $74 + 8 = 82$.

Step 2: Multiply the result in step 1 by 6.

In this example, $82 \times 6 = 492$.

Step 3: Calculate $a \times b$.

In this example, $14 \times 8 = 112$.

Step 4 "Add" them:

$$\begin{array}{ccccc} & 4 & 9 & 2 & \\ + & & 1 & 1 & 2 \\ \hline & 5 & 0 & 3 & 2 \end{array}$$

We get $74 \times 68 = 5032$.

Practice Problems

$23 \times 19 =$	$16 \times 27 =$	$28 \times 17 =$
$26 \times 32 =$	$36 \times 24 =$	$37 \times 28 =$
$41 \times 34 =$	$38 \times 45 =$	$32 \times 43 =$
$53 \times 48 =$	$54 \times 67 =$	$56 \times 42 =$
$51 \times 63 =$	$73 \times 68 =$	$64 \times 74 =$

Math Competition Skill

Divisibility by 9

Digit Sum and Final Digit Sum

Definitions:

The *Digit Sum* of a number is the sum of all digits in the number.

For example, the digit sum of 9876543210 is $9+8+7+6+5+4+3+2+1+0 = 45$. The digit sum of 100,100 is $1+0+0+1+0+0 = 2$.

If the digit sum has two or more digits, we calculate the digit sum for the result again until we obtain one digit. The final one digit is called the *Final Digit Sum*.

The digit sum of 9876543210 is 45. The digit sum of 45 is $4+5=9$. Then 9 is the final digit sum of 9876543210.

Example 1

Find the final digit sum of 8989898989898989.

Answer: 1

Solution:

$$8+9+8+9+8+9+8+9+8+9+8+9+8+9+8+9 = 136,$$
$$1+3+6 = 10,$$
$$1+0 = 1.$$

So 1 is the final digit sum.

Divisibility by 9

Let N be an integer.

How do we test whether N is divisible by 9?

We have the following theorems.

Theorem One

A number is divisible by 9 if and only if the digit sum of the number is divisible by 9.

Example 2

Is 87654321 divisible by 9?

Answer: Yes.

Solution:

Since $8+7+6+5+4+3+2+1 = 36$ is divisible by 9, 87654321 is divisible by 9.

Example 2

Is 789789789789,89 divisible by 9?

Answer: No.

Solution:

The digit sum is

$7+8+9+7+8+9+7+8+9+7+8+9+7+8+9 = 120$.

Since 120 is not divisible by 9, 789789789789 is not divisible by 9.

If we don't know whether 120 is divisible by 9, we can use the theorem again. Since $1+2+0=3$ is not divisible by 9, 120 is not divisible by 9.

Because we can repeat using theorem one, we have the second theorem as follows.

Theorem Two

A number is divisible by 9 if and only if the final digit sum of the number is 9.

The final digit sum cannot be 0. 9 is only one final digit sum such that the original number is divisible by 9.

Example 3

Is 888888888888888888888 divisible by 9?

Answer: No.

Solution:

There are 21 eights in the number. The digit sum is $21 \times 8 = 168$. The digit sum of 168 is $1+6+8=15$. The digit sum of 15 is $1+5=6$. This is the final digit sum of the original number.

So the number given is not divisible by 9.

Deleting Digits

Furthermore, we can simplify the procedure without adding digits.

We delete the digits which are 0s or 9s, or any two or more digits whose sum is 9, 18, etc. Then see what is left. If the original number is divisible by 9, 0 will be left.

Example 4

Is 726394 divisible by 9?

Answer: No.

Solution:

First we delete 9. Then we delete 7 and 2 whose sum is 9, and 6 and 3 whose sum is also 9. Then 4 is left.

Therefore, 726394 is not divisible by 9.

Example 5

Is 8841761993739701954543616000000 divisible by 9?

Answer: Yes.

Solution:

First we delete 0s and 9s.

8,841,761,993,739,701,954,543,616,000,000

Now we delete all pairs of two digits whose sum is 9.

8,841,761, 3,73 ,7 1, 54,543,616, ,

Then we delete all triples of three digits whose sum is 9 or 18.

, 4 ,7 , ,7 ,7 1, , , 1 , ,

Now we have nothing left.

So the original number is divisible by 9.

Proof of the Theorems

Let N be an $(n+1)$-digit number $\overline{a_n a_{n-1} \cdots a_1 a_0}$. Express $N = \overline{a_n a_{n-1} \cdots a_1 a_0}$ in the base 10 expansion:

$$\overline{a_n a_{n-1} \cdots a_1 a_0} = a_n \times 10^n + a_{n-1} \times 10^{n-1} + \cdots + a_1 \times 10 + a_0$$
$$= a_n \times \overbrace{999\cdots9}^{n\ 9\text{'s}} + a_{n-1} \times \overbrace{999\cdots9}^{n-1\ 9\text{'s}} + \cdots + a_1 \times 9$$
$$+ a_n + a_{n-1} + \cdots + a_1 + a_0.$$

Note that $a_n \times \overbrace{999\cdots9}^{n\ 9\text{'s}} + a_{n-1} \times \overbrace{999\cdots9}^{n-1\ 9\text{'s}} + \cdots + a_1 \times 9$ is always divisible by 9.

Therefore, $N = \overline{a_n a_{n-1} \cdots a_1 a_0}$ is divisible by 9 if and only if $a_n + a_{n-1} + \cdots + a_1 + a_0$, the digit sum, is divisible by 9.

This proves theorem one.

Since we can repeat using theorem one, theorem two follows obviously.

Remainder upon Division by 9

Theorem Three

The final digit sum of a number is the remainder of the number upon division by 9.

The proof is obvious.

Example 6

What is the remainder of 86420 upon division by 9?

Answer: 2.

Solution:

The digit sum of 86420 is 20. The final digit sum is $2+0=2$. Then 2 is the remainder.

We can obtain the remainder by deleting the digits as we did in the last section. If we have one digit left, this digit is the remainder. If we have nothing left, the remainder is 0. If we have several digits left, calculate the sum of the left digits. If the sum is one digit, it is the remainder. If this sum is larger than 9, calculate the digit sum of it until you obtain one digit. The one digit is the remainder.

Example 7

What is the remainder of 1234567 upon division by 9?

Answer: 1.

Solution:

2 and 7 make a 9, 3 and 6 make a 9, and 4 and 5 make another 9. Delete them. 1 is left. Then 1 is the remainder.

Example 8

What is the remainder of 2432902008176690000 upon division by 9?

Answer: 5.

Solution:

First we delete 0s and 9s.

2,432,902,008,176,690,000

Then we delete all pairs of two digits whose sum is 9.

2,432, 2, 8,176,6 ,

Now we have the following digits left:

2,4 2, , , ,6 ,

The sum of the left digits is $2+4+2+6=14$. The sum of the digits in 14 is $1+4=5$.

So 5 is the answer.

Examples of Problem Solving

Example 9

$\overline{7m06886}$ is a 7-digit number divisible by 9. Find m.

Answer: 1

Solution:

The digit sum is $7+m+0+6+8+8+6=35+m$.

Note that *m* is a digit. That is, $0 \le m \le 9$. The only value for *m* is 1 such that $35+m$ is divisible by 9.

Example 10

$\overline{1a2b34}$ is a 6-digit number yielding remainder 8 upon division by 9. Find the sum of all possible values for $a+b$.

Answer: 23

Solution:

The digit sum is $1+a+2+b+3+4=10+a+b$.

Note that *a* and *b* are digits. So $0 \le a \le 9$ and $0 \le b \le 9$. Then $0 \le a+b \le 18$.

If $a+b=7$, the digit sum is 17. The remainder is the final digit sum: $1+7=8$.

If $a+b=16$, the digit sum is 26. The remainder is the final digit sum: $2+6=8$.

No other value for $a+b$ exists such that the property is satisfied.

Therefore, the answer is $7+16=23$.

Example 11

$N=\overline{a8888888b8}$ is a 10-digit number divisible by 72 where *a* and *b* are digits with $a>0$. Find all possible values of *a*.

Answer: 4, 8 and 9

Solution:

Note that $72=8\times 9$. Since 8 and 9 are relatively prime, *N* must be divisible by both 8 and 9.

For *N* to be divisible by 8, $\overline{8b8}$ must be divisible by 8. So $b=0$, 4 or 8.

If $b=0$, the digit sum of *N* is $64+a$. For *N* to be divisible by 9, $64+a$ must be divisible by 9. Thus $a=8$.

If $b=4$, the digit sum of *N* is $68+a$. Then $a=4$.

If $b=8$, the digit sum of *N* is $72+a$. We have $a=0$ or 9. Since $a>0$, $a=9$.

Therefore, there are three possible values of *a*: 4, 8, and 9.

Practice Problems

1. Circle the numbers divisible by 9:

   234 2215 11609 23301 2229 70718 75161 33 21213 60606 10119

2. Find the remainder for each upon division by 9:

   11111 1234 987 1001 2008 12121212 97531 222 20081105 3456

3. Is 1124000727777607680000 divisible by 9?

4. When 6604484017332394393600000 is divided by 9, what is the remainder?

5. $22\cdots 2$ in which there are *k* 2's is divisible by 9. What is the minimum value of *k*?

6. $\overline{123,4m4,3215}$ is a 9-digit number divisible by 9. Find *m*.

7. I repeat 12345 *m* times to form a large number:

   $$\overline{123451234512345\cdots 12345},$$

   If the number is divisible by 9, find the possible smallest value of *m*.

8. I wrote all natural numbers from 1 to *n* ($n>10$) together to form a large integer:

   $$\overline{123456789101112\cdots n}.$$

   If the number is divisible by 9, what is the possible smallest value of *n*?

9. $N=\overline{a1,111,111,11b}$ is an 11-digit number which is a multiple of 72, where *a* and *b* are digits with $a>0$. Find $a-b$.

A Problem from a Real Math Competition

Today's problem is from University of Northern Colorado Mathematics Contest (UNCMC).

(UNCMC 2000-2001 First Round Problem 11)

35! has 41 digits when written out and can be printed as a diamond read row by row. What is the missing center digit?

```
            1
         0  3  3
      3  1  4  7  9
   6  6  3  8  6  1  4
4  9  2  9  □  6  6  6  5
   1  3  3  7  5  2  3
      2  0  0  0  0
         0  0  0
            0
```

Answer: 6

Solution:

35! must be divisible by 9.

Why?

Thus the sum of all digits must be divisible by 9. With the following steps we can obtain the answer.

1. Add all 40 digits.
2. Find the remainder when the sum is divided by 9. Or find the final digit sum.
3. Subtract the remainder from 9.

The difference is the answer.

The shortcut to get the answer is to delete digits.

First delete all 0's and 9's. Then the table becomes

				1				
				3	3			
		3	1	4	7			
	6	6	3	8	6	1	4	
4		2		□	6	6	6	5
	1	3	3	7	5	2	3	
		2						

Then delete all pairs of two digits whose sum is 9. Then the table may look

			1				
						1	
4				□			
	1						3
		2					

Then delete all triples of three digits whose sum is 9. Then we have

		1			
					1
			□		
1					

Three 1's are left, whose sum is 3.

Therefore, the missing number at the center must be $9-3=6$.

Practice Problem

25! has 36 digits. We truncate the ending 6 zeros and write the remaining 25 digits in some order in the 5 by 5 array. What is the missing digit at the center?

8	8	4	1	7
1	9	5	4	6
0	1	□	5	1
7	6	3	4	9
9	3	7	3	9

Answers to All Practice Problems in Last Issue

Mental Calculation

Practice Problems I

8979012	8967018	8958049
8961036	8973020	8973008
8994001	8982009	8976016
8949042	8934117	8934105
8913208	8922165	8901272

Practice Problems II

15964020	24980003	35946018
48881072	63928014	80901028
15572312	24401199	35328535
47492772	62354710	79138807
15178815	23951211	34733342
47513413	62165340	78969870

Counting Squares

Practice Problems I

1. 30	2. 85	3. 51
4. 55	5. 50	6. 66
7. 55	8. 378	9. 130

Practice Problems II

1. 55	2. 16	3. 91
4. $\frac{16}{9}$	5. 67	6. 91
7. 14		

A Problem from a Real Math Competition

41

Solutions to Creative Thinking Problems 52 to 54

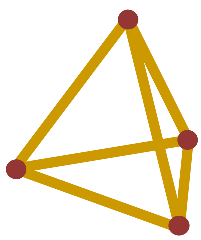
52. Make One Word

My answer is

"one word".

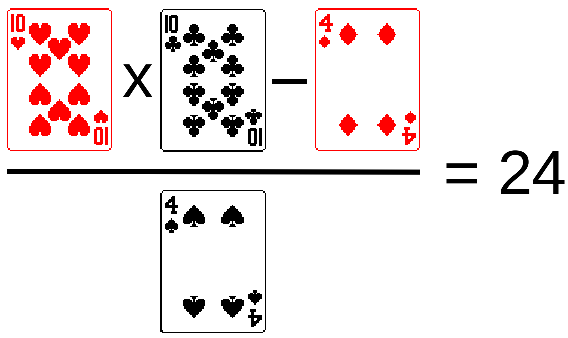
53. Make 4 Equilateral Triangles

Use the six matchsticks to make a regular tetrahedron, which has four equilateral triangles as faces:



54. Another Challenge to Make 24

My solution is:



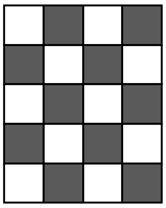
Clues to Creative Thinking Problems 55 to 57

55. 3 x 3 Matrix

Pay attention to the number at the center of the matrix. The sum of the nine numbers is 9 times that number.

56. Covering with Tetrominos

Let us color the 4×5 rectangle with the standard chessboard coloring:



57. Weighing Meat II

Study from small numbers.

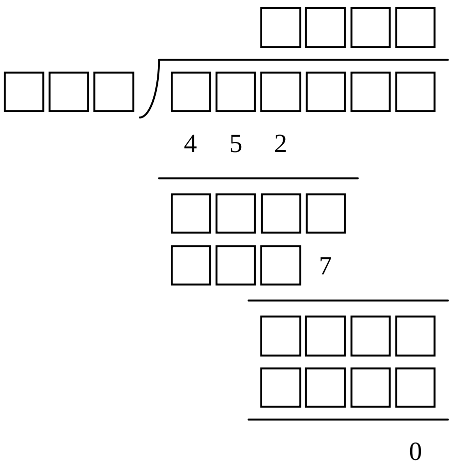
Again we need a weight of 1 pound for one pound of meat. For a piece of meat of 2 pounds, we may place the meat with the one-pound weight together. We can make the next weight heavier.

Now you continue.

Creative Thinking Problems 58 to 60

58. A Division

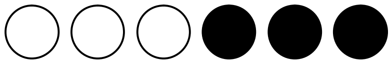
Fill the blanks with digits such that the following division expression is true.



59. Moving Checkers

There are three white checkers and three black checkers in a line, as shown below:

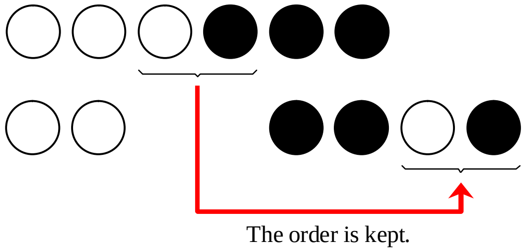
Move them such that the white and black checkers are lined alternatively.



Rules:

1. Move any two consecutive checkers onto the empty space of the same line;
2. Don't change the order of the two checkers when they are being moved;

For example, the move shown is allowed:

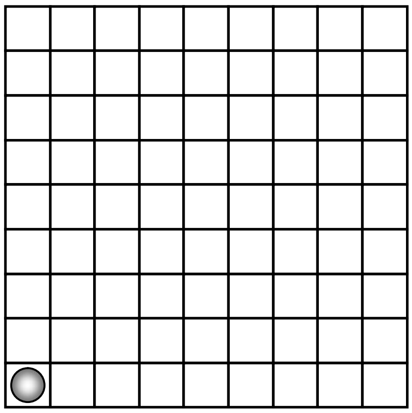


However, it is not the proper first move.

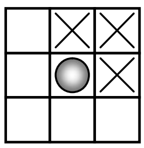
3. You are allowed to move checkers three times.

60. A Checkerboard Game

A 9×9 checkerboard is shown below. A checker is at the left-bottom corner. Let us play a game. We take turns to move the checker.



We are allowed to move the checker one square right, or up, or right-up only. That is, the checker in the figure below can be moved only to one of the squares marked by X.



Whoever moves the checker to the right-top corner is the winner.

Do you want to go first (of course to win)? Is there a winning strategy for any one?

(Clues and solutions will be given in the next issues.)

